The Case for a Positive Cosmological Λ-term

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Abstract: Recent observations of Type 1a supernovae indicating an accelerating universe have once more drawn attention to the possible existence, at the present epoch, of a small positive Λ-term (cosmological constant). In this paper we review both observational and theoretical aspects of a small cosmological Λ-term. We discuss the current observational situation focusing on cosmological tests of Λ including the age of the universe, high redshift supernovae, gravitational lensing, galaxy clustering and the cosmic microwave background. We also review the theoretical debate surrounding Λ: the generation of Λ in models with spontaneous symmetry breaking and through quantum vacuum polarization effects – mechanisms which are known to give rise to a large value of Λ hence leading to the 'cosmological constant problem'. More recent attempts to generate a small cosmological constant at the present epoch using either field theoretic techniques, or by modeling a dynamical Λ-term by scalar fields are also extensively discussed. Anthropic arguments favouring a small Λ-term are briefly reviewed. A comprehensive bibliography of recent work on Λ is provided.

1. Introduction

Recent years have witnessed a resurgence of interest in the possibility that a positive Λ-term (a cosmological constant) may dominate the total energy density in the universe. Interest in the cosmological constant stems from several directions:

(i) Observations of high redshift Type 1a supernovae appear to suggest that our universe may be accelerating with a large fraction of the cosmological density in the form of a cosmological Λ-term. When combined with observations of the cosmic microwave background (CMB), an approximately flat Friedmann-Robertson-Walker (FRW) cosmological model with total energy density ($\Omega_m + \Omega_\Lambda \simeq 1$) is suggested, in agreement with predictions of the simplest versions of the inflationary scenario of the early universe (sections 4.3, 4.4).

(ii) Most dynamical estimates of the amount of clustered matter yield a conservative upper limit $\Omega_m \lesssim 0.3$. In addition, theoretical modelling of structure formation based on the cold dark matter model (CDM) with $\Omega_m = 1$ has failed to match up with observations at a quantitative level. By contrast, a flat low density CDM+Λ universe with $\Omega_m \simeq 0.3$ and $\Omega_\Lambda \simeq 0.7$, and with an approximately flat (or, Harrison-Zeldovich-like, $n_S \approx 1$) initial Fourier spectrum of scalar (adiabatic) inhomogeneous metric and density perturbations agrees remarkably well with a wide range of observational data ranging from large and intermediate angle CMB anisotropies to observations of galaxy clustering on large scales. Since an approximately flat initial spectrum of adiabatic perturbations is also precisely what simplest variants of the inflationary scenario predict, the positive Λ-term removes a necessity in any complications of the inflationary scenario (which might be required if the universe was found to be open).

(iii) At a theoretical level, a cosmological constant $\Lambda = 8\pi G\rho_{vac}/c^2$ is predicted to arise out of zero-point quantum vacuum fluctuations of fundamental scalar, spinor, vector and tensor fields (see section 5). Although a theoretically predicted value of ρ_{vac} usually appears to be much larger than current observational limits, there is no generic known mechanism which will set the value of Λ to precisely zero either on the basis of symmetry arguments or by dynamical means. [1] Some recent attempts to generate a small Λ at the present epoch either through vacuum polarization and particle creation effects or by means of dynamically evolving scalar fields are discussed in section 7.

[1] The value of Λ can, of course, be set to zero by hand by adding suitable counterterms to the bare (infinite) value of Λ in the Lagrangian. This method, however, amounts to a rather ad-hoc adjustment of parameters and cannot be regarded as being 'generic' (see section 5).

Although none of the above arguments can by themselves be regarded as conclusive evidence for a cosmological constant, the growing body of work on the subject, combined with a possible deep relationship between a small cosmological constant today and a large cosmological term driving inflation at an early epoch, suggests that the case for a positive cosmological constant be taken seriously. In this paper we attempt to review some aspects of the cosmological constant issue emphasizing both theoretical as well as observational aspects. For earlier reviews on the subject the reader is referred to Zeldovich (1968), Weinberg (1989) and Caroll, Press and Turner (1992).

From the physical point of view, a Λ-term represents a new type of dark non-baryonic matter, completely unknown from laboratory experiments. Its difference from another type of dark non-baryonic matter that has been already introduced in cosmology for almost two decades from observations of gravitational clustering is essentially that matter described by the Λ-term is, (a) not gravitationally clustered at all scales at which we see clustering of baryons and dustlike dark matter, and (b) has a strongly negative effective pressure ($P < 0,\ |P| \sim \rho c^2$). Thus, remarkably, by investigating the behaviour of the present universe we are studying novel fundamental physics. Extragalactic astronomy and cosmology once more become a driving force for new insights in physics !

2. The Cosmological Constant revisited

In 1917, only a few years after introducing the field equations of the General theory of relativity (GR), Einstein proposed adding a 'cosmological constant' to these equations which were modified to

$$R_{ik} - \frac{1}{2}\, g_{ik}R - \Lambda g_{ik} = \frac{8\pi G}{c^4} T_{ik}. \qquad (1)$$

The main motivation behind introducing the cosmological constant appears to have been Einstein's belief that the equations of General Relativity should be compatible with Mach's principle. Einstein was fascinated by the arguments of philosopher/scientist Ernst Mach. Mach was concerned about the notion of absolute motion which prevailed in Newtonian mechanics. He postulated that all the matter in the universe including the distant stars provided a 'background' against which motion could be measured and that unless there was a material background which served as a reference frame, it was meaningless to talk of rest or motion in any absolute sense (Mach 1893). Einstein proposed incorporating Mach's principle into the general theory of relativity by suggesting a solution of the equations (1) in which the universe was static and closed on itself, much like the closed two dimensional surface of a balloon. A static solution of (1) is possible to construct since, as shown in section 3, a positive cosmological constant introduces a repulsive force which can counterbalance the attractive force of gravity leading to the 'static Einstein universe'. This universe has a finite spatial volume with no boundaries, furthermore the total mass in such a universe is related directly to its (finite) volume (section 3.1). A low mass universe has a small volume, and an empty universe has no volume at all ! The static Einstein universe thus incorporates Mach's principle since it demonstrates that without matter there can be no space against which background inertial effects can be measured.

It should be borne in mind that in 1917 the idea of the Milky Way being an island universe was widely believed in, and the notion of the existence of other galaxies had not yet been firmly established. All this was about to change however, when in the early 1920s Slipher's work showed that light from several spiral nebulae (later re-christened galaxies) was redshifted, a fact that could be explained by the Doppler effect if these nebulae/galaxies were moving away from us. [2] In 1922, about five years after Einstein had proposed his static solution, Aleksander Friedmann constructed a matter dominated expanding universe without a cosmological constant. The possibility that the universe may be expanding led Einstein to abandon the idea of a static universe and, along with it, the cosmological constant. In a 1923 letter to Weyl, Einstein is quoted as saying [150] " If there is no quasi-static world, then away with the cosmological term !" The conclusive discovery by Hubble (1929) of a linear expansion law relating redshift to distance made Friedmann models the standard geometrical framework within which Hubble's discoveries were subsequently interpreted [205,213,142,154].

Introduced, then discarded, the cosmological constant staged several comebacks, the first having to do with the realization that the static Einstein universe was unstable and, if perturbed, could either expand or contract. In 1927 Lemaitre constructed an expanding model which originated from such an asymptotically static state in the distant past. The Lemaitre model had a long age and has frequently been reinvoked whenever the age constraints (associated with high values of H_0) get too tight for standard FRW models (section 4.1). The Lemaitre model was also discussed in the early 1960s when observations appeared to show an excess of quasi-stellar objects (QSO's) near the redshift $z \simeq 2$. It was felt that a universe which 'hesitated' or 'loitered' near the quasi-static state at $z \sim 2$ for a sufficient amount of time would

[2] It is interesting that the same year that Einstein introduced the cosmological term Λ, de Sitter presented solutions of (1) with $T_{ik} = 0$, $\Lambda > 0$, which had both static and dynamic features. Intriguingly, although the space-time coordinatization originally introduced by de Sitter was static [43], namely $ds^2 = \cosh^{-2} Hr[dt^2 - dr^2 - H^{-2}\tanh^2 Hr(d\theta^2 + \sin^2\theta d\phi^2]$, it allowed for a linear redshift-distance relation, since $\Gamma^r_{tt} \neq 0$ in the above metric resulting in the motion of test bodies by virtue of the geodesic equation $\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$ (Γ^i_{kl} is the affine connection). This effect was pointed out by Weyl (1923) and later used by Eddington to interpret Slipher's observations in the context of de Sitter's static universe [205].

naturally explain an abundance of objects at that redshift. Present arguments for a positive cosmological constant are associated with observations of high redshift supernovae which indicate $\Omega_\Lambda = \Lambda/3H^2 \sim 0.7$, and from cosmological simulations of structure formation which also appear to favour a positive cosmological constant [114,144]. In the next section we shall qualitatively analyze solutions of the Einstein equations with a non-zero cosmological constant in a Friedmann-Robertson-Walker (FRW) universe following the original path taken by Eddington and Lemaitre.

3. FRW Cosmological models with $\Lambda \neq 0$.

A homogeneous and isotropic universe is characterized by the Friedmann-Robertson-Walker line element

$$ds^2 = c^2dt^2 - a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) \qquad \kappa = 0, \ \pm 1 \tag{2}$$

In this metric the Einstein equations (1) with matter in the form of a perfect fluid acquire the following simple form

$$3(\frac{\dot{a}}{a})^2 = 8\pi G\rho + \Lambda c^2 - 3\frac{\kappa c^2}{a^2}, \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2) + \frac{\Lambda c^2}{3}. \tag{4}$$

Equation (4) can be recast to look like the equation of motion of a point particle on the surface of a sphere of radius $R \equiv a$ and mass M, setting $c = 1$ we obtain

$$\ddot{R} = -\frac{GM}{R^2} + \frac{\Lambda}{3}\,R. \tag{5}$$

The total 'gravitating mass' $M = \frac{4\pi}{3}R^3(\rho+3P)$ reflects the fact that 'pressure carries weight' in Einstein's theory of gravity. From (5) we find that a particle on the sphere feels both attractive and repulsive forces. The force of repulsion $F_{rep} = \frac{\Lambda}{3}\,R$ is caused by the cosmological constant and increases with distance if $\Lambda > 0$. (For negative Λ this becomes a force of 'attraction', formally resembling the force of confinement between quarks which binds them within the nucleus.)

The opposite signs of the forces of attraction and repulsion in (5) allow for a large number of new solutions to the Einstein equations. As pointed out in the previous section, Einstein himself used the repulsive effect of the cosmological constant to balance the attraction of matter resulting in a static closed universe which Einstein felt was in agreement with Mach's principle. A quantitative analysis of solutions to (3) & (4) can be gained by eliminating ρ in these equations and combining them into a single equation for the evolution of the scale factor in the presence of a Λ-term

$$2\frac{\ddot{a}}{a} + (1+3w)[\frac{\dot{a}^2}{a^2} + \frac{\kappa c^2}{a^2}] - (1+w)\Lambda c^2 = 0 \tag{6}$$

which is also valid if Λ is a function of time (*i.e.* if $T_{ik} = \Lambda(t) g_{ik}$). (We have assumed that matter has an equation of state $P = w\rho c^2$.) A comprehensive quantitative analysis of (6) has been carried out in [62] for a cosmological constant, and in [147] for a time varying cosmological term $\Lambda(t)$. For our purpose it will be sufficient to note that the qualitative behaviour of the universe in the presence of a cosmological term which is either constant or time varying, can be understood very simply by rewriting (3) in the suggestive form (we assume $c = 1$ for simplicity)

$$\frac{1}{2}\dot{a}^2 + V(a) = E \tag{7}$$

where

$$V(a) = -(\frac{4\pi G}{3}\rho a^2 + \frac{\Lambda a^2}{6}), \quad E = -\frac{\kappa}{2}. \tag{8}$$

Since $\rho = \rho_0 (a_0/a)^{3(1+w)}$, we find, substituting $w = 0$ for dust

$$V(a) = -(\frac{A}{a} + \frac{\Lambda a^2}{6}) \tag{9}$$

where $A = \frac{4\pi G}{3}\rho_0 a_0^3$. (We assume for simplicity that matter is pressureless so that $w = 0$, however the qualitative analysis given below remains valid for matter possessing more general equations of state.) Equation (7) reminds one of classical motion with conserved energy E in a one dimensional potential $V(a)$ whose generic form is shown in Figure 2 for $w = P/\rho \geq 0$. ¿From the form of $V(a)$ several things can be said about the behaviour of the expansion factor $a(t)$. We shall first examine the case $\kappa = 1$ ($E < 0$) since it provides us with the largest variety of qualitatively different solutions to the Einstein equations.

3.1. Closed universe models ($\kappa = 1$).

Consider a particle moving with negative total energy under the influence of the potential $V(a)$ shown in fig. (2), then the following situations arise (the one dimensional particle coordinate is equivalent to the value of 'a' – the expansion factor.)

(i) *Oscillating models*: The particle moves from left to right (starting from $a = 0$) but with insufficient energy to surmount the potential barrier. Consequently the expansion factor $a(t)$ first increases then decreases describing a universe which, after expanding, contracts into a singularity. Such models are called *oscillating models of the first kind* [142].

(ii) *Bouncing models*: The particle moves from right to left (starting from $a = \infty$) again with insufficient energy to surmount the potential, in this case $a(t)$ first decreases then increases and the universe rebounds after collapsing without ever reaching a singular state. Such models are called *bouncing models* or *oscillating models of the second kind*, an example of such a model is provided by the complete de Sitter space-time

$$ds^2 = c^2dt^2 - H^{-2}\cosh^2(Ht)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] \tag{10}$$

where $-\infty < t < \infty, 0 \le \chi \le \pi, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$.

(iii) *Static Einstein Universe (SE)*: The particle is placed at the top of the potential with exactly zero kinetic energy: $\ddot{a} = \dot{a} = 0$. This situation, describes the static Einstein universe. Setting $\ddot{a} = \dot{a} = 0, \kappa = 1$ in (3) and assuming for simplicity that matter is pressureless ($w = 0$) we obtain

$$\Lambda_{crit} = \frac{4\pi G}{c^2}\rho_m = \frac{1}{a_0^2} \tag{11}$$

which relates the value of the cosmological constant to the density of matter *and* the curvature of space. The volume and mass of a SE universe are respectively $V = 2\pi^2 a_0^3$, $M = V\rho_m = 2\pi^2 a_0^3\rho_m$. As a result $M = (\frac{\pi c^2}{2G})a_0$, and one finds $\lim_{a_0\to 0} M \simeq 0$, *i.e.* the mass of the static Einstein universe decreases as its radius shrinks to zero, consequently a static empty universe simply cannot exist ! This feature of SE found favour with the proponents of Mach's principle as discussed in section 2.

(iv) *Loitering Universe*: The static Einstein universe is clearly unstable: small fluctuations can make it either contract or expand (these correspond to tiny perturbations of a particle located at the hump of $V(a)$ in fig. (2) which cause it to roll either towards the left ($a \to 0$) or towards the right ($a \to \infty$). Based on this observation, an interesting new model of the universe was proposed by Eddington and Lemaitre in which the value of Λ was kept slightly larger than Λ_{crit}. In this case the universe begins from the Big Bang, approaches the static Einstein universe and remains close to it for a substantial period of time before re-expanding [53,122]. (If $\Lambda < \Lambda_{crit}$ the universe will contract instead of expanding.) The quasi-static or *loitering* phase, during which the universe remains close to $a \simeq a_0$, has several appealing features not present in models which expand monotonically [169]: (i) density perturbations grow at the exponential (Jeans) rate $\delta \propto \exp\sqrt{4\pi G\rho}\, t$ and not at the weaker rate $\delta \propto t^{2/3}$ characteristic of an Einstein-de Sitter universe; (ii) a prolonged quasi-static phase results in an older universe, ameliorating the 'age' problem which can arise in matter dominated flat cosmologies if the value of the Hubble parameter turns out to be large (see section 4.1).

Interest in loitering models rose dramatically in the late 1960's when observations suggested the existence of an excess of quasars near redshift $z_l \simeq 2$. To explain these observations the Lemaitre model with a quasi-static (loitering) phase at $z_l \simeq 2$ was invoked [160,175,166]. (Loitering at z_l arises if the cosmological constant exactly balances ρ_m leading to the relation: $(1+z_l)^3 = \Omega_\Lambda/\Omega_m$, where $\Omega_\Lambda = \Lambda/3H^2$. A decaying cosmological constant will lead to loitering at higher values of z_l which has certain advantages from the standpoint of current observations [169].)

(v) *Monotonic Universe*: The particle approaches the potential from the left ($a = 0$) with sufficient energy to surmount it and travel on towards $a \to \infty$. In such a situation the scale factor will have an inflection point at $\ddot{a} \simeq 0$, $\dot{a} > 0$. By adjusting initial conditions so that the particle remains close to the hump of the potential for a sufficiently long duration, one recovers the 'loitering' models discussed in (iv).

(vi) *Nonsingular Oscillating model*: Another cosmological model deserving mention consists of a form of matter which behaves as a Λ-term when the universe is small, as the universe expands the Λ-term decays into either radiation or matter. The energy density in such a model can be phenomenologically described by $8\pi G\rho = \Lambda/(1+\Lambda a^p/\alpha)$, so that $\lim_{a\to 0} 8\pi G\rho \simeq \Lambda$, $\quad \lim_{a\to\infty} 8\pi G\rho \simeq \alpha/a^p$, $p = 3,4$ for matter and radiation respectively. The potential $V(a) = -\frac{4\pi G}{3}\rho a^2$ associated with this model has a broad minimum which leads to a non-singular oscillatory motion of the expansion factor $a(t)$. This toy model is interesting since it exhibits an infinite number of expansion and contraction cycles without ever becoming singular.

(vii) Other possibilities not shown in Figure (1) include 'asymptotic models' in which the universe asymptotically approaches or moves away from the static Einstein universe. The reader is referred to [62,142] for a more quantitative discussion of these issues.

Although the above discussion referred to cosmological models filled with matter having non-negative pressure and a cosmological constant, it is easy to show that the qualitative behaviour of the universe described in (i) – (vii) remains valid, if we generalize the definition of the Λ-term to include any form of matter which violates the *strong energy condition* so that $\rho_\Lambda + 3P_\Lambda < 0$ [169].

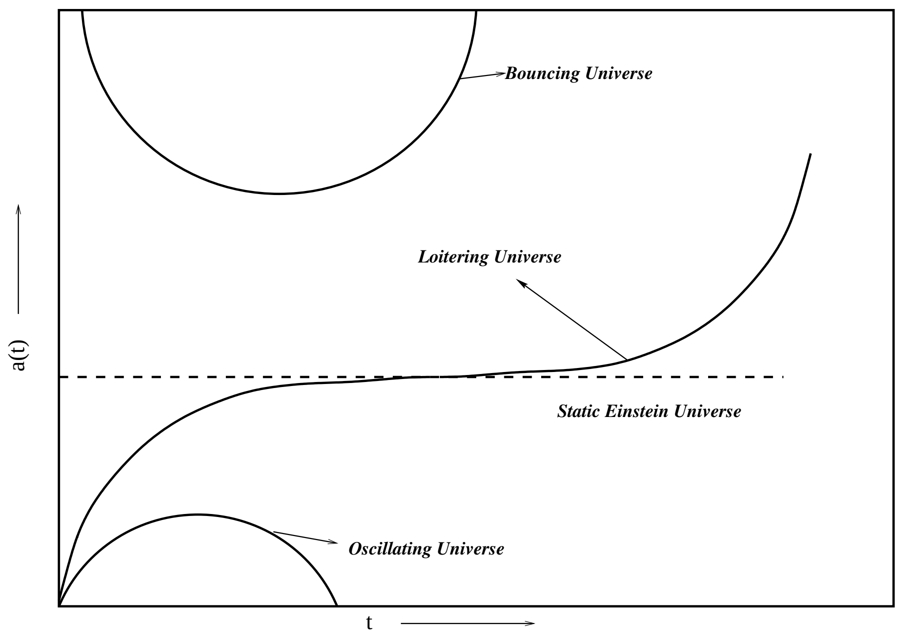

Fig. 1. Four distinct possible solutions of the Einstein equations with a cosmological constant are schematically shown for a closed universe ($\kappa = +1$). (Incidentally none of these solutions arise if $\kappa = 0, -1$.)

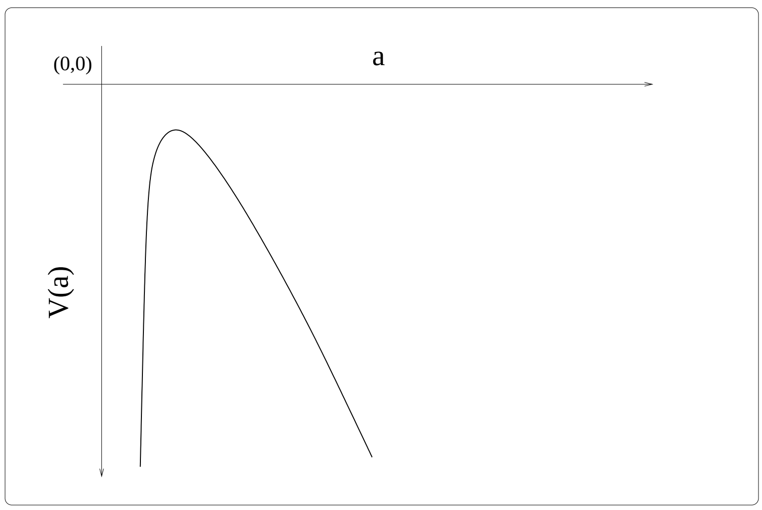

Fig. 2. The 'effective potential' $V(a)$ describing the expansion of the universe in the presence of matter and a cosmological constant (see equation (7)). The large variety of solutions to the Einstein equations can be analyzed by studying the kindered problem of the motion of a particle moving under the influence of the potential V.

3.2. Spatially open and flat cosmological models.

The preceding discussion referred to closed universe models for which $\kappa = 1$ and $E < 0$. For flat and open models ($\kappa = 0, -1$) the total energy is non-negative $E \geq 0$ and motion in the potential $V(a)$ becomes unbounded, since a particle always has sufficient energy to surmount the potential barrier in figure (2). As a result the expansion factor $a(t)$ shows monotonic behaviour, starting from the singular point at $a = 0, t = 0$ and increasing without bound as $t \to \infty$. For $\Lambda > 0$ the universe passes through an inflection point at which the expansion of the universe changes from deceleration ($\ddot{a} < 0$) to acceleration ($\ddot{a} > 0$) (from (3) & (4) it can be shown that this usually occurs at a redshift when Λ is still not dominating the expansion dynamics of the universe; see section 4.3).

In the important case when the universe is spatially flat and contains pressureless matter (dust) and a positive cosmological constant, the expansion factor has the exact analytical form:

$$a(t) \propto \left(\sinh \frac{3}{2}\sqrt{\frac{\Lambda}{3}}ct\right)^{2/3} \tag{12}$$

which interpolates smoothly between the matter dominated epoch in the past ($a \propto t^{2/3}$) and an inflationary epoch in the future ($a \propto e^{\sqrt{\Lambda/3}t}$). Equation (12) will be used later, when we examine some observational aspects of a universe with a cosmological constant in Section 4

Finally, oscillating, bouncing and loitering models, as well as the static Einstein universe, are clearly absent in flat and open FRW models.

4. Observational consequences of a cosmological Λ-term

Arguments favouring $\Lambda > 0$ at the present epoch essentially stem from four sets of observations:

(i) *The age issue:* A high value of the Hubble constant $H_0 \sim 80$km/sec/Mpc predicts a short age of the universe which is incompatible with the ages of the oldest stars (12 - 16 Gyr) unless the universe is open ($\Omega_m < 0.1$) or flat and Λ dominated $\Omega_m + \Omega_\Lambda = 1$. The appeal of this argument has somewhat decreased following recent Hipparcos parallax measurements indicating a lower value $H_0 \leq 67$ km/sec/Mpc and also a lower age for globular clusters: 11.5 ± 1.5 Gyr. Still, recent observations of old galaxies at high redshifts are extremely difficult to accommodate within the framework of a flat matter dominated cosmology unless the Hubble parameter is very small ($H_0 \lesssim 45$ km/sec/Mpc; section 4.1).

(ii) *Structure formation:* The standard COBE normalized cold dark matter model of structure formation with $\Omega_m = 1$ appears to be in serious conflict with observations. The situation may be remedied if the universe is flat, with most of matter smoothly distributed in the form of a cosmological constant and only a small fraction $\Omega_m h \simeq 0.2$ in clustered matter. (Here h is the Hubble constant in units of 100 km/s/Mpc). Studies of the abundance and evolution of clusters of galaxies and of lensing by clusters also appear to favour a low density universe (section 4.6).

(iii) *Baryon excess in clusters:* In a spatially flat universe with $\Omega_m = 1$ the mass fraction in baryons in the Coma cluster is expected to greatly exceed nucleosynthesis bounds leading to what has been called the 'baryon catastrophe'. The mass fraction in baryons can be kept in agreement with nucleosynthesis constraints only if $\Omega_m h \simeq 0.16$ [210] (Ω_m includes contribution from baryons and clustered dark matter). Agreement with the Inflationary scenario which strongly favours a spatially flat universe then suggests that the remaining mass might be in the form of a cosmological constant.

(iv) *High redshift supernovae and the cosmic microwave background:* Preliminary results from this rapidly advancing field of cosmology suggest that the universe may be accelerating universe with a dominant contribution to its energy density coming in the form of cosmological Λ-term. These results, when combined with CMB anisotropy observations on intermediate angular scales, strongly support a flat universe $\Omega_m + \Omega_\Lambda = 1$ with $\Omega_\Lambda \sim 0.6 - 0.7$ (sections 4.3 & 4.4).

In the first half of this paper we shall briefly review the present observational status of the cosmological constant referring the reader to the original papers and earlier reviews [26,37] for more details.

4.1. H_0, q_0 and the Age of the Universe

The quest for understanding the geometry of our universe has been one of the central aims of cosmology since the 1960s and Alan Sandage in 1970 even described the whole of observational cosmology as being a "search for two numbers". The first of these numbers – the Hubble parameter $H_0 = (\dot{a}/a)_0$, provides us with measure of the observable size of the universe and its age. The second $q_0 = -H_0^{-2}(\ddot{a}/a)_0$ is called the deceleration parameter and probes the equation of state of matter and the cosmological density parameter. In the presence of a cosmological constant,

$$q_0 = \frac{\Omega_m}{2} - \Omega_\Lambda. \tag{13}$$

In a critical density universe with $\Omega_m + \Omega_\Lambda = 1$, the deceleration parameter

$$q_0 = \frac{3}{2}\Omega_m - 1, \tag{14}$$

consequently a critical density universe will accelerate if $\Omega_m < 2/3$. The observational quest for q_0 showed that evolutionary effects play a dominant role in this important quantity and for a while it was felt that it may be virtually impossible to disentangle the true cosmological 'signal' for q_0 from evolutionary 'noise'. Recent years however have witnessed an important turnaround with the development of new and more powerful techniques which are either less sensitive to evolutionary effects or for which evolutionary effects are better understood.

In the next section, we shall consider several promising cosmological tests which could shed light on the composition of the universe and its geometrical properties. These tests include gravitational lensing, the use of high redshift supernovae as calibrated standard candles, and the angular size-redshift relation. Before we do that however, we shall turn our attention to another fundamental quantity which has traditionally played an important role in constraining cosmological models – the age-redshift relation.

The presence of a Λ-term leads to an increase in the age of the universe with far-reaching observational consequences. To appreciate this let us first consider the critical density Einstein-de Sitter universe with $a \propto t^{2/3}$, so that

$$t_0 = \frac{2}{3}H_0^{-1}. \tag{15}$$

The value of H_0 therefore serves to determine the age uniquely in a spatially flat matter dominated universe. Moderately high values $H_0 \geq 75$ km s^{-1} Mpc^{-1} result in an age for the universe which is smaller than the ages of the oldest globular clusters making an Einstein-de Sitter universe with a high value of H_0 difficult to reconcile with observations. The situation can be remedied if we live in an open universe. Assuming for simplicity that the universe is empty (a good approximation if $\Omega_m \leq 0.2$) we get $a \propto t$ so that

$$t_0 = H_0^{-1}. \tag{16}$$

Combining (15) and (16) we get $\frac{2}{3}H_0^{-1} \leq t_0 < H_0^{-1}$ for matter dominated cosmological models with $\Omega_m \leq 1$. (A longer age $t_0 > H_0^{-1}$ can be achieved in the presence of a cosmological constant.) An open universe though older, nevertheless has two difficulties associated with it: the first is related to the growth of density perturbations which slow down considerably in an open universe leading to large primordial fluctuations in the Cosmic Microwave Background which may be difficult to reconcile with observations (assuming standard adiabatic fluctuations with scale-invariant initial spectra). The second is related to the Omega problem: a low Omega universe requires extreme fine tuning of initial conditions, which some find to be an unattractive feature of open/closed models.

Let us now consider a more general situation in which the universe has a Λ-term in addition to normal matter. A closed form expression for the age of the universe in spatially flat models is given by [117]

$$t_0 = \frac{2}{3H_0}\Big[\frac{1}{2\ \Omega_\Lambda^{1/2}} \log\frac{1+\Omega_\Lambda^{1/2}}{1-\Omega_\Lambda^{1/2}}\Big] \tag{17}$$

where $\Omega_\Lambda = \Lambda/3H_0^2 = 1-\Omega_m$.

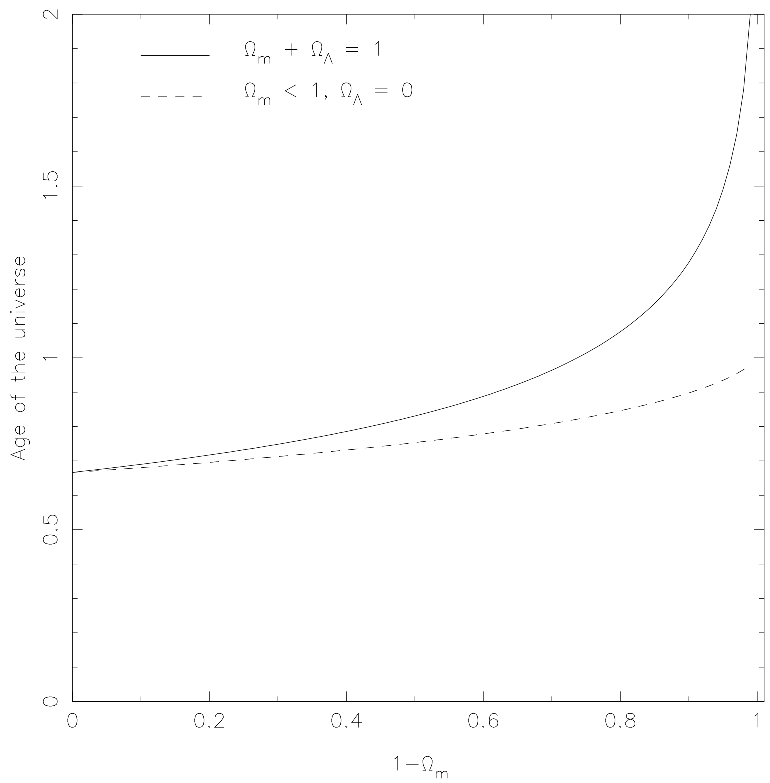

Fig. 3. The age of the universe (in units of H_0^{-1}) is shown as a function of $1-\Omega_m$ for (i) flat models with a cosmological constant $\Omega_m + \Omega_\Lambda = 1$ (solid line), and (ii) for open cosmological models $\Omega_m < 1$ (dashed line).

In Figure 3 we show the present age of a universe consisting of matter and a cosmological constant and parametrized in terms of the variables Ω_Λ and Ω_m.

We find that the age of a flat universe with $\Omega_\Lambda = 1 - \Omega_m$ is always greater than that of an open universe for identical values of $1 - \Omega_m$. Additionally t_0 can exceed H_0^{-1} if $\Omega_\Lambda \gtrsim 0.74$.

No exact forms for $t(H)$ are available for a time dependent Λ-term. To study this and other cases, it is useful to express the Hubble parameter as a function of the cosmological redshift z. This can easily be done for a general multicomponent universe consisting of several non-interacting matter species characterized by equations of state $P_\alpha = w_\alpha \rho_\alpha$, for which the Hubble parameter can be written as

$$H(z) = H_0 h(z) = H_0(1+z)\left[1 - \Omega_{total} + \sum_\alpha \Omega_\alpha (1+z)^{\gamma_\alpha}\right]^{\frac{1}{2}}. \tag{18}$$

where $\Omega_{total} = \sum_\alpha \Omega_\alpha$, $\gamma_\alpha = 1 + 3w_\alpha$ and $1 + z = a_0/a(t)$ is the cosmological redshift parameter.

Let us assume that the universe, in addition to matter and radiation, consists of a decaying Λ-term modelled by a fluid with equation of state $P_X = (m/3 - 1)\rho_X$ so that $\Lambda = \Lambda_0(a_0/a)^m$, $m \leq 2$. The dimensionless Hubble parameter $h(z)$ then becomes

$$h(z) = \left[(1 - \Omega_{total})(1+z)^2 + \Omega_m(1+z)^3 + \Omega_\Lambda(1+z)^m\right]^{\frac{1}{2}} \tag{19}$$

where $m = 0$ corresponds to a cosmological *constant*, and we neglect the presence of radiation. In a spatially flat universe $\Omega_{total} = \Omega_\Lambda + \Omega_m = 1$ (the present value of Λ is therefore given by $\Lambda_0 = 3H_0^2[1 - \Omega_m]$). A useful relationship between the cosmological time parameter t and the cosmological redshift z can be obtained by differentiating $1 + z = a_0/a(t)$ with respect to time, so that $dz/dt = -H(z)(1+z)$. This leads to the following completely general expression for the age of the universe at a redshift z

$$t(z) = H_0^{-1} \int_z^\infty \frac{dz'}{(1+z')h(z')}. \tag{20}$$

with $h(z)$ supplied by either (18) or (19).

A running debate over the previous decade or so has centered around whether or not the universe has an 'age problem', *i.e.* on whether matter dominated cosmological models are substantially younger than their oldest constituents (which happen to be metal poor old globular cluster stars). A key role in this controversy is played by the Hubble parameter, whose present value is known to within an uncertainty of about two. Higher values of H_0 clearly give rise to a younger universe whereas lower values lead to an older one.

At the time of writing lower values $H_0 \lesssim 65$ km s^{-1}Mpc^{-1} are strongly supported by observations, especially in the light of new parallax measurements made by the *Hipparcos* satellite for Cepheid stars, which has led to a reanalysis of distances to globular clusters and consequently of their age estimates [3] which have dropped to 11.5 ± 1.5 Gyr [30,120]. (Low values of H_0 are also suggested from an analysis of the Sunyaev-Zeldovich effect from X-ray emitting clusters [99], from Type 1a supernovae[165,158] and from Cepheids observed by the HST.) Lower values of H_0 reconcile matter dominated flat models with the revised ages of globular clusters [120] and with limits from nucleochronology which indicate $t_0 \geq 7.8$ Gyr [30]. Low values of H_0 combined with an absence of stellar systems with ages greatly exceeding 20 Gyr also argue against large values of Λ, since from Fig 3 we see that $\Omega_\Lambda \gtrsim 0.85$ suggests an age $t_0 \gtrsim 24$ Gyr (if $H_0 = 50$km/sec/Mpc). Finally the recent supernovae based measurements of Perlmutter et al. (1998b) suggest a best-fit age of the universe $t_0 \simeq 14.9\,(\frac{63}{H_0})$ Gyr for a spatially flat universe with $\Omega_m \simeq 0.28$ and $\Omega_\Lambda \simeq 0.72$.

The above arguments were largely limited to the *present* age of the universe. Ages of high redshift objects at $z > 1$ provide crucial information about the age of the universe at that redshift [110]. The existence of at least two high redshift galaxies having an evolved stellar population and hence an old age sets very severe constraints on a flat matter dominated universe [49,119,50,152]. For instance the radio galaxy 53W091 at $z = 1.55$ discovered by Dunlop et al. (1996) is reported to be at least 3.5 Gyrs old. The age of a spatially flat matter dominated universe at a redshift z is easily obtained from (20) to be

$$t(z) = \frac{2}{3H_0}(1+z)^{-\frac{3}{2}}. \tag{21}$$

Consequently the discovery of 53W091 can be accommodated within an $\Omega_m = 1$, CDM model only if the Hubble parameter is uncomfortably small [119] $H_0 \lesssim 45$ km s^{-1}Mpc^{-1}. However both open and flat Λ-dominated models alleviate the age problem for 53W091.

At even higher redshifts, recent work [212] aimed at age-dating a high redshift QSO at $z = 3.62$ using delayed iron enrichment by Type Ia supernovae as a cosmic clock, sets a lower bound of 1.3 Gyr on the age of the universe at that redshift. This discovery can be accommodated within a spatially flat cosmology only if $\Omega_m + \Omega_\Lambda = 1$ (low density open models with $\Omega_m \leq 0.2$ are also permitted). However, the age dating of stellar populations requires complex modelling and although both open and flat Λ-dominated models are clearly favoured by current observations, more work needs to be done before matter dominated flat models are excluded on the basis of age arguments alone. [4]

[3] An important indicator of the absolute age of a globular cluster star is its luminosity when it leaves the main sequence. Since luminosity is related to distance (to the star) ages of globular clusters are very sensitive to distance callibrators.

[4] It may be appropriate to mention that models with a cosmological constant may *never*

4.2. The luminosity distance and gravitational lensing

Before proceeding to discuss possible constraints on Ω_Λ from gravitational lensing in this section and high redshift supernovae in the next, let us introduce a quantity which plays a crucial role in these discussions, namely the *luminosity distance* $d_L(z)$ upto a given redshift z. Consider an object of absolute luminosity $\mathcal{L}$ located at a coordinate distance r from an observer at $r = 0$. Light emitted by the object at a time t is received by the observer at $t = t_0$, t and t_0 being related by the cosmological redshift $1 + z = a(t_0)/a(t)$. The luminosity flux reaching the observer is

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2}, \tag{22}$$

where d_L is the luminosity distance to the object [142]

$$d_L = a(t_0)r(1 + z). \tag{23}$$

The luminosity distance d_L depends sensitively upon both the spatial curvature and the expansion dynamics of the universe. To demonstrate this we determine d_L using the expression for the coordinate distance r obtained by setting $ds^2 = 0$ in (2), resulting in

$$\int_0^r \frac{dr'}{\sqrt{1 - \kappa r'^2}} = \int_t^{t_0} \frac{dt}{a(t)} = \eta_0 - \eta \tag{24}$$

which gives

$$\begin{aligned} r &= \sin\ (\eta_0 - \eta) \qquad (\kappa = +1) \\ &= \qquad \eta_0 - \eta \qquad (\kappa = 0) \\ &= \sinh\ (\eta_0 - \eta) \qquad (\kappa = -1) \end{aligned} \tag{25}$$

where $\eta = \int_0^t c dt/a(t)$.

Furthermore, since $dz/dt = -(1 + z)H(z)$, we get

$$\eta_0 - \eta = \int_t^{t_0} \frac{cdt}{a(t)} = \frac{c}{a_0 H_0} \int_0^z \frac{dz'}{h(z')} \tag{26}$$

be singular and therefore could possess an infinite age as demonstrated by the 'bouncing models' in Fig. 1. However the value of the cosmological constant in such models is several orders of magnitude larger than permitted by current observations. The relevance of such models is therefore likely to be limited to the very early universe and will not affect the age problem discussed here.

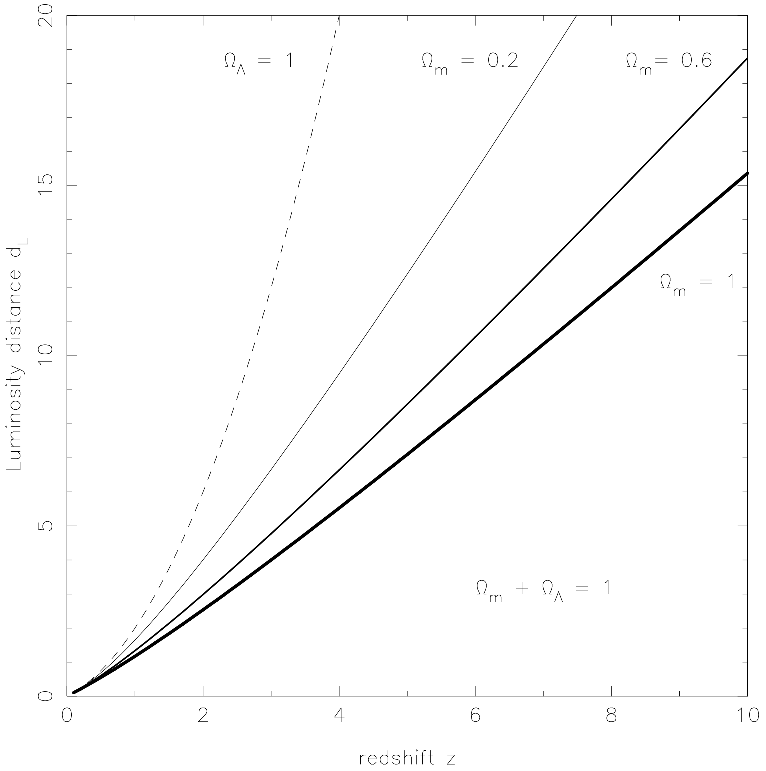

Fig. 4. The luminosity distance d_L (in units of H_0^{-1}) is shown as a function of cosmological redshift z for flat cosmological models with a cosmological constant $\Omega_m + \Omega_\Lambda = 1$. Heavier lines correspond to larger values of Ω_m. For comparison we also show (dashed line) the angular size in a flat de Sitter universe ($\Omega_\Lambda = 1$).

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where $h(z) = H(z)/H_0$ is defined in (18), and, in a universe with several components

$$\frac{\kappa}{a_0^2 H_0^2} = \Omega_{total} - 1. \tag{27}$$

Substituting (27) and (26) in (23) we get the following expression for the luminosity distance in a multicomponent universe with a cosmological term [26]

$$d(z) = \frac{(1+z)cH_0^{-1}}{|\Omega_{total} - 1|^{\frac{1}{2}}} S(\eta_0 - \eta) \tag{28}$$

where

$$\eta_0 - \eta = |\Omega_{total} - 1|^{\frac{1}{2}} \int_0^z \frac{dz'}{h(z')}, \tag{29}$$

and $S(x)$ is defined as follows: $S(x) = \sin(x)$ if $\kappa = 1$ ($\Omega_{total} > 1$), $S(x) = \sinh(x)$ if $\kappa = -1$ ($\Omega_{total} < 1$), $S(x) = x$ if $\kappa = 0$ ($\Omega_{total} = 1$).

Before we turn to applications, let us consider a simple example which provides us with an insight into the role played by the luminosity distance d_L in cosmology. In a spatially flat universe the expression for d_L simplifies considerably, so that we get for the matter dominated model ($a \propto t^{2/3}$)

$$d_L^{MD} = 2c\ H_0^{-1}\{(1+z) - (1+z)^{\frac{1}{2}}\}, \tag{30}$$

on the other hand in de Sitter space ($a \propto \exp{(H_0 t)}$)

$$d_L^{DS} = c\ H_0^{-1} z(1+z). \tag{31}$$

Comparing (30) and (31) we find $d_L^{DS}(z) > d_L^{MD}(z)$, which means that an object located at a fixed redshift will appear brighter in an Einstein-de Sitter universe than it will in de Sitter space (equivalently in the steady state model).[5] This is also true for a two component universe consisting of matter and a cosmological constant as demonstrated in Fig 4. In a spatially flat universe the presence of a Λ-term increases the luminosity distance to a given redshift, leading to interesting astrophysical consequences. Since the physical volume associated with a unit redshift interval increases in models with $\Lambda > 0$, the likelihood that light from a quasar will encounter a lensing galaxy is larger in such models. Consequently the probability that a quasar is lensed by intervening galaxies increases appreciably in a Λ dominated universe, and can be used as a test to constrain the value of Ω_Λ [72,71,192]. Following [73,26,37] we give below the probability of a quasar at redshift z_s being lensed relative to the fiducial Einstein-de Sitter model ($\Omega_m = 1$)

$$P(\text{lens}) = \frac{15}{4}[1 - \frac{1}{\sqrt{1+z_s}}]^{-3} \int\limits_0^{z_s} \frac{(1+z)^2 dz}{h(z)} \left[\frac{d(0,z)d(z,z_s)}{d(0,z_s)}\right]^2 \tag{32}$$

where $d(z_1, z_2)$ is a generalization of the angular distance $d_A = d_L(1+z)^{-2}$ discussed in Section 4.5:

$$d(z_1, z_2) = \frac{1}{(1+z_2)|\Omega_{total} - 1|^{\frac{1}{2}}} S(\eta_{12}) \tag{33}$$

where

$$\eta_{12} = \eta_1 - \eta_2 = |\Omega_{total} - 1|^{\frac{1}{2}} \int\limits_{z_1}^{z_2} \frac{dz}{h(z)} \tag{34}$$

and $S(\eta_{12})$ is defined as follows, $S(\eta_{12}) = \sin{(\eta_{12})}$ if $\kappa = 1$ ($\Omega_{total} > 1$), $S(\eta_{12}) = \sinh{(\eta_{12})}$ if $\kappa = -1$ ($\Omega_{total} < 1$), $S(\eta_{12}) = \eta_{12}$ if $\kappa = 0$ ($\Omega_{total} = 1$). In Fig 5 we show the lensing probability $P(\text{lens})$ for the spatially flat universe $\Omega_m + \Omega_\Lambda = 1$. A large increase in the lensing probability over the fiducial $\Omega_m = 1$ value is clearly seen in models with low Ω_m (high Ω_Λ). (For a broader analysis of parameter space see [26].)

[5] For instance a galaxy at redshift $z = 3$ will appear 9 times brighter in a flat matter dominated universe than it will in de Sitter space (see Fig 4).

Turning now to the observational situation, at the time of writing the best observational estimates give a 2σ upper bound $\Omega_\Lambda < 0.66$ obtained from multiple images of lensed quasars [111,112,136]. Since radio sources are not plagued by some of the systematic errors arising in an optical search (notably extinction in the lens galaxy and the quasar discovery process) a search involving radio selected lenses can yield useful complementary information to optical searches [60]. Recent work by Falco et al (1998) gives $\Omega_\Lambda < 0.73$ which is only marginally consistent with optical estimates, a combined analysis of optical and radio data yields a slightly more conservative upper bound $\Omega_\Lambda < 0.62$ at the 2σ level (for flat universes) [60]. (Constraints on Ω_Λ from both lensing and Type 1a Supernovae are discussed in [201]; also see next section. An interesting new method of constraining Ω_Λ from weak lensing in clusters is discussed in [67], also see [12] and section 4.6.) Improved understanding of statistical and systematic uncertainties combined with new surveys and better quality data promise to make gravitational lensing a powerful technique for constraining cosmological parameters and cosmological world models.

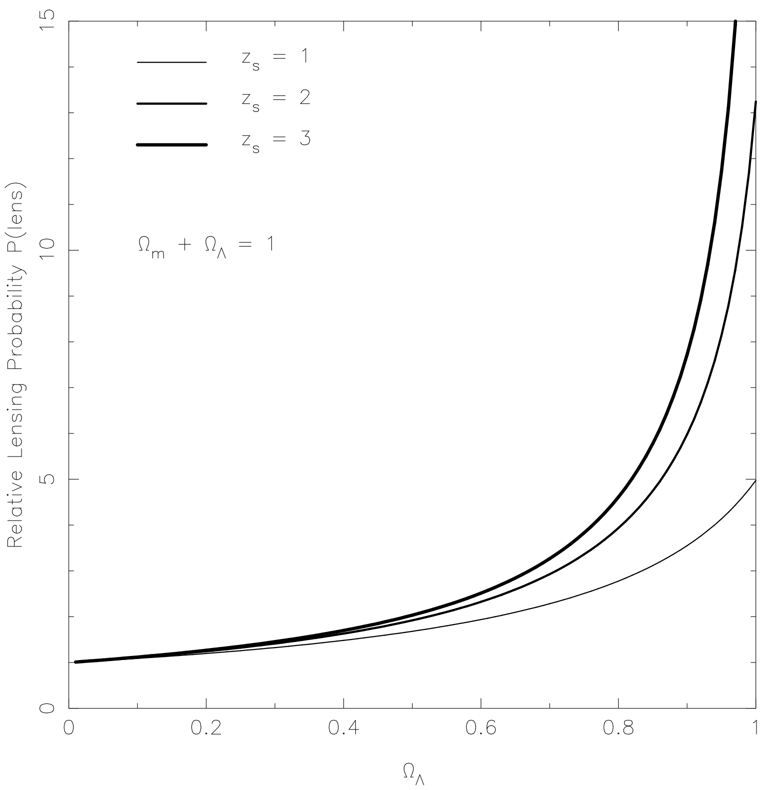

Fig. 5. The lensing probability $P(lens)$ evaluated relative to the fiducial case $\Omega_m = 1$ is shown as a function of Ω_Λ for flat cosmological models $\Omega_m + \Omega_\Lambda = 1$. The source redshift is taken at $z_s = 1, 2, 3$ respectively.

4.3. Type 1a Supernovae and the value of Λ.

The luminosity distance also plays a crucial role in determining cosmological parameters once the absolute brightness of a class of objects is known. Of considerable importance in this context is the magnitude-redshift relation which relates the apparent magnitude m of an object to its absolute magnitude M

$$\mu \equiv m - M = 5 \log_{10} \frac{d_L}{Mpc} + 25 \tag{35}$$

where μ is known as the distance modulus. Since d_L depends upon the geometry of space and its material content, the magnitude-redshift relation (35) can, in principle be used to determine Ω_Λ and Ω_{total} if both m and M are known within reasonable limits. [6]

The recent discovery that type Ia supernovae may be used as calibrated standard candles for obtaining estimates of the luminosity distance d_L through (35) has aroused great interest. Type Ia supernovae are explosions which arise as a white dwarf star crosses the Chandrasekhar stability limit while accreting matter from a companion star [94,4,40]. The high absolute luminosity of SNe Ia ($M_B \simeq -19.5$ mag) suggests that they can be seen out to large distances making them ideal candidates for measuring and constraining cosmological parameters [39,21]. Of crucial import to using type Ia supernovae for estimating the luminosity distance d_L has been the observation that: (i) the dispersion in their luminosity at maximum light is extremely small ($\lesssim$ 0.3 mag); (ii) the width of the supernova light curve is strongly correlated with its intrinsic luminosity: a brighter supernova will have a broader light curve indicative of a more gradual decline in its brightness [161]. Both (i) and (ii) reduce the scatter in the absolute luminosity of type 1a supernovae to $\sim 10\%$ making them excellent standard candles [21].

Nearby type 1a supernovae have been used to determine the value of H_0 whereas those further away are used to obtain reasonable estimates of cosmological parameters by minimizing the χ^2 statistic

$$\chi^2(H_0, \Omega_m, \Omega_\Lambda) = \sum_i \frac{\{\mu_{p,i}(z_i; H_0, \Omega_m, \Omega_\Lambda) - \mu_{0,i}\}^2}{\sigma^2_{\mu 0,i}} \tag{36}$$

where μ_p are model dependent 'predicted' values of the distance modulus obtained from (28) and (35), and $\mu_0(z_i)$ are the observed values. At least two

[6] In practice (35) must be corrected for effects associated with the redshifting of light as it travels to us, commonly called the K-correction. For instance photons being detected using a red filter would originally have had a 'blue spectrum' if the source was located at $z \simeq 1$. Other possible sources of systematic errors include luminosity evolution, intergalactic extinction, Malmquist bias, the aperture correction, weak lensing etc. A more complete discussion of these issues can be found in [142,154].

groups – the Supernova Cosmology Project [157] and the High-Z Supernova Search Team [165] have been engaged in both finding and calibrating supernovae at low and high redshifts. At the time of writing, both groups have analyzed data for several dozen type Ia supernovae and a consensus seems to be emerging that a positive value of Ω_Λ is strongly preferred. For instance, treating type 1a supernovae as standard candles and then using distance estimates to 42 moderately high redshift supernovae with $z \lesssim 0.83$, Perlmutter et al. (1998b) find that the joint probability distribution of the parameters Ω_Λ & Ω_m is well approximated by the relationship (valid for $\Omega_m \leq 1.5$)

$$0.8\Omega_m - 0.6\Omega_\Lambda \simeq -0.2 \pm 0.1.$$

The best-fit confidence region in the $\Omega_m - \Omega_\Lambda$ plane shown in (6) appears to favour a closed universe. However, as we shall see in the next section, when combined with the results of cosmic microwave experiments, the combined likelihood of $\Omega_m,\ \Omega_\Lambda$ peaks near $\Omega_m + \Omega_\Lambda \simeq 1$.

These results provide an interesting insight into the expansion dynamics of the universe during its recent past. For instance, a cosmological model which passed through an epoch of matter domination before the present Λ dominated epoch, also passed through an inflexion point at which the expansion of the universe changed from deceleration ($\ddot{a} < 0$) to acceleration ($\ddot{a} > 0$). From (3) & (4) it can be shown that this occurred at a redshift when Λ was still not dominating the expansion dynamics of the universe. For instance from (4) we find that deceleration is succeeded by acceleration at the epoch

$$(1 + z_*)^3 = 2\frac{\Omega_\Lambda}{\Omega_m}. \tag{37}$$

On the other hand the epoch of equality between ρ_m and Λ occurred at the redshift

$$(1 + z_\star)^3 = \frac{\Omega_\Lambda}{\Omega_m} \tag{38}$$

where $\Omega_\Lambda = \Lambda c^2/3H_0^2$. Substituting the 'best-fit' values obtained by Perlmutter et al. (1998b) for a flat universe $\Omega_m \simeq 0.28$, $\Omega_\Lambda \simeq 0.72$ we get $z_* \simeq 0.726$ and $z_\star \simeq 0.37$ so that $z_\star < z_*$. From (14) we also get $q_0 = -0.58$ for the deceleration parameter, indicating an accelerating universe (the combined Sn+CMB data give a slightly larger value $q_0 \simeq -0.5$).

Supernovae data can also be used to constrain time dependent Λ models of the kind discussed in section 8. In the case of scalar field models with potentials $V(\phi) \sim \phi^{-p}$ and $V(\phi) \sim (e^{1/\phi} - 1)$ the scalar field density in a spatially flat universe is constrained to lie in the range [159,203] $\Omega_\phi \in (0.6, 0.7)$ and the effective equation of state is $w_\phi < -0.6$ (at the 95% confidence level).

A combined analysis of gravitational lensing and Type 1a supernovae gives the best-fit value $\Omega_m \simeq 0.33$ for a spatially flat universe [201]. Attempts to

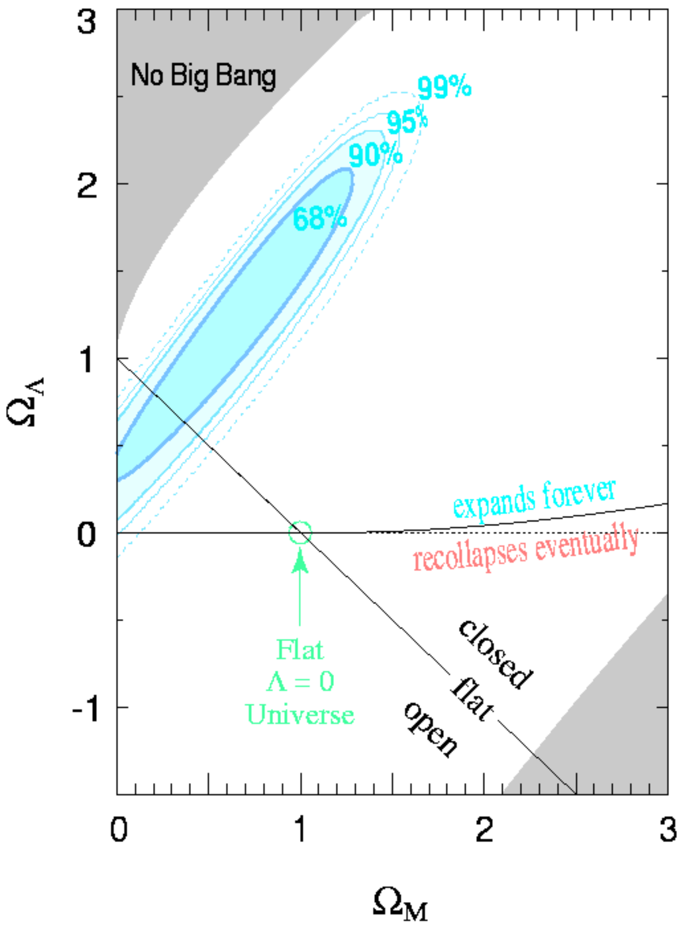

Fig. 6. Best-fit confidence regions in the $\Omega_m - \Omega_\Lambda$ plane obtained from the analysis of Type 1a high redshift supernovae of Perlmutter et al. (1998b). The upper-left shaded region corresponds to the singularity free 'bouncing universe' models discussed in section 3.1.

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constrain the decay rate of a time-dependent cosmological term $\Lambda = \Lambda_0(1+z)^m$ result in $0.24 \lesssim \Omega_m \lesssim 0.38$ and $m \lesssim 0.85$ (at the 68% confidence level) which in turn places constrains on the cosmic equation of state $w = m/3 - 1 \lesssim -0.72$. The combined supernovae & lensing data therefore convincingly rule out a network of tangled cosmic strings ($w \simeq -1/3$) and strongly favour a cosmological *constant* ($w = -1$).

The results obtained by both the Supernova Cosmology Project and the High-Z Supernova Search Team team present the strongest 'direct' evidence for a non-zero cosmological constant. However much work needs to be done both in understanding systematic uncertainties as well as Sn Ia properties before the case for a positive Λ is firmly established.[7] As we shall show in the next section, much stronger constraints on Ω_m and Ω_Λ emerge if we combine the supernovae results with observations of the cosmic microwave background.

4.4. Constraints on Λ from the cosmic microwave background.

On large angular scales $\theta \gtrsim 1^\circ$ photons of the cosmic microwave background traveling to us from the last scattering surface probe scales that were causally unconnected at the time of recombination.[8] As a result observations of the CMB anisotropy on large scales provide us with a very clean probe of the primordial matter fluctuation spectrum before its distortion by astrophysical processes. On such large scales the main contribution to the CMB anisotropy comes from the Sachs-Wolfe effect

$$\frac{\delta T}{T} = -\frac{1}{2}\int_{\eta_{rec}}^{\eta_0} \frac{\partial h_{\alpha\beta}}{\partial \eta} e^\alpha e^\beta d\eta, \tag{39}$$

which relates temperature fluctuations to the integral of the variation of the metric evaluated along the line of sight [167]. The evaluation of (39) in a flat matter dominated universe is simplified by the fact that linearized the gravitational potential does not evolve with time, with the result that the above expression reduces to

$$\frac{\delta T}{T} \simeq \frac{1}{3}\frac{\delta \phi}{c^2} \tag{40}$$

which relates fluctuations in the CMB to those in the gravitational potential at the surface of last scattering. Equation (40) can therefore be successfully used to determine the amplitude of primordial metric fluctuations with the help of COBE data. The presence of a cosmological constant however causes the linearized gravitational potential to evolve with time, the full Sachs-Wolfe integral (39) must therefore be used both to determine and normalize the primordial fluctuation spectrum [113].

The CMB temperature distribution can be written as

$$T(\theta,\phi) = T_0\left[1 + \frac{\delta T}{T}(\theta,\phi)\right], \tag{41}$$

where T_0 is the blackbody temperature $T_0 = 2.728 \pm 0.004^\circ K$ [65]. $\delta T/T$ can

[7] An accelerating universe can also be accommodated within the framework of the Quasi-Steady State Cosmology of Hoyle, Burbidge and Narlikar (1993).

[8] In matter dominated models the horizon at last scattering subtends an angle $\theta \simeq 1.8^\circ \Omega_m^{1/2}(1000/z_{rec})^{1/2} \simeq 1.8^\circ$ for $\Omega_m \simeq 1$ and $z_{rec} \simeq 1000$. In flat Λ dominated models the dependence of θ on Ω_m is much weaker, consequently $\theta \simeq 1.8^\circ$ provides a good approximation for most values of Ω_m.

be written in terms of a multipole expansion on the celestial sphere:

$$\frac{\delta T}{T}(\theta,\phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_l^m(\theta,\phi), \tag{42}$$

Information pertaining to a particular theoretical model is contained in the coefficients a_{lm} which are usually assumed to be statistically independent and distributed in the manner of a Gaussian random field with zero mean and variance

$$C_l \equiv \langle |a_{lm}|^2 \rangle \tag{43}$$

where the angle brackets indicate an ensemble average over possible universes.

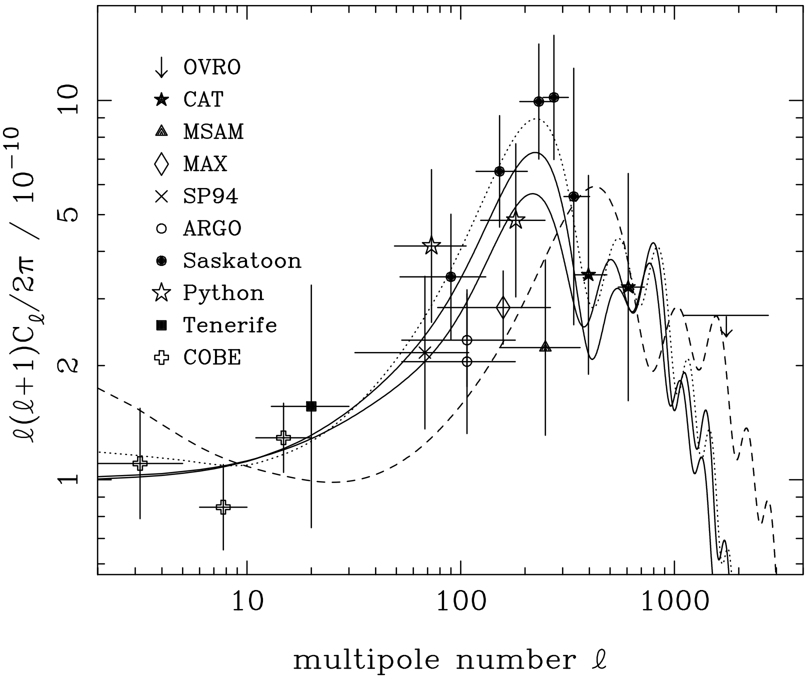

Fig. 7. The angular power spectrum of the cosmic microwave background is plotted against the angular wavenumber l (in radians^{-1}). The predictions of the following theoretical models are tested against observations: (i) The flat ΛCDM model with parameters $(\Omega_\Lambda, \Omega_m, \Omega_b, h) = (0.7, 0.3, 0.05, 0.65)$ (dotted line); (ii) Flat CDM models with $(\Omega_m, \Omega_b, h) = (1, 0.1, 0.5)$ and $(\Omega_m, \Omega_b, h) = (1, 0.05, 0.5)$ (solid lines), the larger Ω_b model shows a higher Doppler peak; (iii) Open CDM model with $(\Omega_m, \Omega_b, h) = (0.3, 0.05, 0.65)$ (broken line). Here $\Omega_m = \Omega_{cm} + \Omega_b$, where Ω_{cm} is the cold (non-baryonic) matter component. For more details, see Peacock (1999) and Bond et al. (1997).

The quantity that is directly measured by observations is the angular correlation of the temperature anisotropy

$$C(\theta) = \left\langle \frac{\delta T}{T}(\hat{n_1})\frac{\delta T}{T}(\hat{n_2})\right\rangle = \frac{1}{4\pi}\sum_l \left[\frac{l+\frac{1}{2}}{l(l+1)}\right] C_l P_l(\cos\theta) W_l \qquad (44)$$

where $\cos\theta = \hat{n_1}\cdot\hat{n_2}$, P_l are Legendre polynomials and W_l is the filter function of the experiment used to measure the CMB; $\langle\rangle$ denote an ensemble average in the case of theoretical predictions and angular average in the context of observations. (The relationship between $C(\theta)$ and the angular power spectrum C_l is analogous to that between the two point correlation function ξ and the matter power spectrum $P(k)$.)

At low multipoles $l \lesssim 60$ the contribution to C_l is mainly from the Sachs-Wolfe effect due to scalar density perturbations and (in some models) tensorial gravity waves. (The value of the tenth multipole provides a convenient choice for normalization of the perturbation spectrum [22].) At large $l > 60$ however, the main contribution to C_l is due to oscillations in the photon-baryon plasma before decoupling, which leave their imprint in the CMB at the time of last scattering. These oscillations give rise to Doppler peaks in C_l the location of the peak being determined by the angle subtended by the sound horizon at the time of recombination (see figure 7). The sound horizon depends upon $\Omega_{\rm baryon}$ & Ω_m whereas the angular diameter distance to the last scattering surface depends upon Ω_Λ, Ω_m and the spatial curvature of the universe. (Both Ω_Λ and the spatial curvature are extremely small at the time of last scatter and therefore do not contribute to the sound horizon. On the other hand, the location of the doppler peak is not very sensitive to Ω_{baryon} provided $\Omega_{baryon} \ll \Omega_m + \Omega_\Lambda$.)

Since the angular scale corresponding to the first Doppler peak is sensitive to both the curvature of the universe and its matter content, its location can be used to place strong constraints on cosmological models. There are some indications that the first Doppler peak has been measured near $l \simeq 260$ [86]. (The height of the peak is related to the baryon fraction in the universe and also to the scalar/tensor ratio S/T, the larger the baryon density the higher the peak, a small value of S/T reduces the peak height. The peak height also depends on the rate of expansion of the universe and hence on H_0 [97]; for low values $\Omega_{\rm baryon} \lesssim 0.05$ the peak height decreases if H_0 increases, whereas the reverse is true for a larger baryon fraction.) In figure 7 we show the angular power spectrum of the cosmic microwave background for the flat ΛCDM model with $\Omega_\Lambda = 0.7$ (dotted line), for comparison we also show spatially flat (solid line) and open (dashed line) matter dominated models with $\Omega_m = 1$ and $\Omega_m = 0.3$ respectively.

It should however be pointed out that the CMB alone cannot uniquely differentiate between two models having identical matter content, perturbation

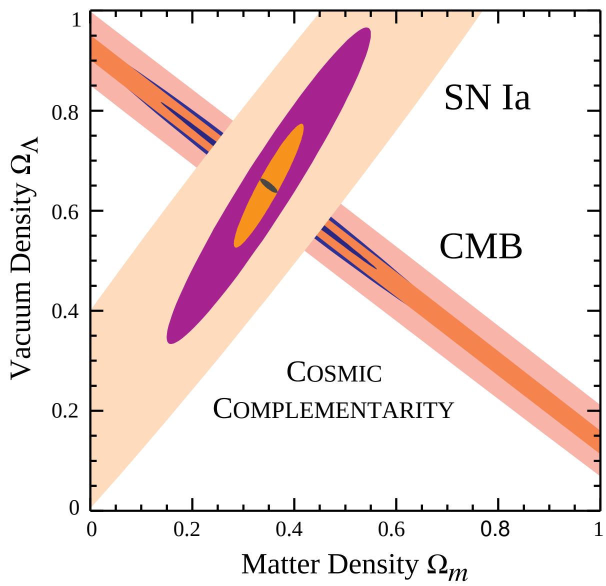

Fig. 8. The 'cosmic complementarity' principle is beautifully illustrated by these best-fit contours obtained using expected data from future supernovae and CMB experiments. The 68% confidence regions are shown for three sets of hypothetical supernovae data likely to be recorded in five years time. The CMB analysis refers to the upcoming MAP and PLANCK satellite missions. The assumed fiducial model is ΛCDM with $\Omega_m = 0.35$, $\Omega_\Lambda = 0.65$ and $H_0 = 65$ km. sec^{-1} Mpc^{-1}. One clearly sees that the degeneracy in parameter space from supernovae observations is almost orthogonal to the degeneracy arising from CMB measurements. For more details see Tegmark et al. (1998).

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spectra and with the same angular diameter distance to the last scattering surface. Such models will be degenerate in the sense that they will produce very similar CMB anisotropies [56,57]. A degeneracy in parameter space happens to be a common feature of most cosmological tests. Fortunately different tests often have complementary degeneracies. (A degeneracy arises when a result remains unaffected by a specific combination of parameter changes.) For instance the degeneracy in the $\Omega_m - \Omega_\Lambda$ plane from high redshift supernovae tests is almost orthogonal to that in a CMB analysis. Thus combining Type 1a supernovae measurements with the results from CMB experiments can serve to substantially decrease the errors on expected values of Ω_m and Ω_Λ as illustrated in figure 8 and figure 9 [209,190,56,57]. Since the location of the Doppler peak near $l \simeq 260$ supports a spatially flat universe [86], a combined likelihood analysis of CMB anisotropy and Type 1a Supernovae data gives the best fit values [57]

$$\Omega_m = 0.25^{+0.18}_{-0.12}, \;\; \Omega_\Lambda = 0.63^{+0.17}_{-0.23}. \tag{45}$$

which strongly favour a flat universe with $\Omega_m + \Omega_\Lambda \simeq 1$ (also see [130,191]).

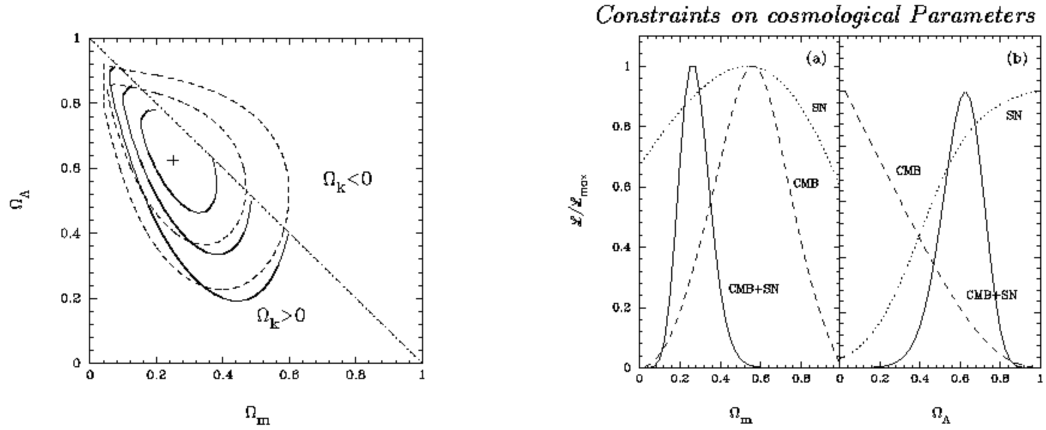

Fig. 9. Likelihood contours in the $\Omega_m - \Omega_\Lambda$ plane (left) are derived using a combined likelihood analysis of CMB and supernovae data. These contours show that the combined CMB+Sn likelihood function is strongly peaked at $\Omega_m = 0.25$ and $\Omega_\Lambda = 0.63$ thereby favouring a flat universe (shown by the dotted-dashed straight line). The marginalized likelihood functions on the right are shown for SN data alone (dotted lines), CMB data alone (dashed lines) and the combined SN and CMB data (solid lines). The CMB+SN likelihood function sharply peaks near $\Omega_m + \Omega_\Lambda = 1$. More details may be found in Efstathiou et al. (1998).

4.5. The Angular size - redshift relation.

Another potentially sensitive test of models is related to the fact that the angular size $\Delta\theta$ of an extended object D located at a redshift z, depends rather sensitively on the properties of the cosmological model in which it is being measured. Knowing the absolute size of an object (e.g. galaxy or radio source) and the angle subtended by a distribution of such objects in the universe, it may be possible (after correcting for projection and evolution effects) to say something about the geometry of space and the matter content of the universe.

It is easy to derive a relationship between D and $\Delta\theta$. Consider an object of proper length D at a coordinate distance r, and assume for simplicity that the object is aligned along the θ axis so that coordinates marking its 'top' and 'bottom' are respectively $(r, \theta_1 + \Delta\theta_1, \phi_1)$ and (r, θ_1, ϕ_1). The observer is at $r = 0$. The proper length of the object can be obtained by setting $t =$ constant in the FRW line-element (2) giving [142]

$$ds^2 = -D^2 = -a^2(t) r^2 \Delta\theta_1^2. \tag{46}$$

As a result we get the following expression for the angle subtended by the

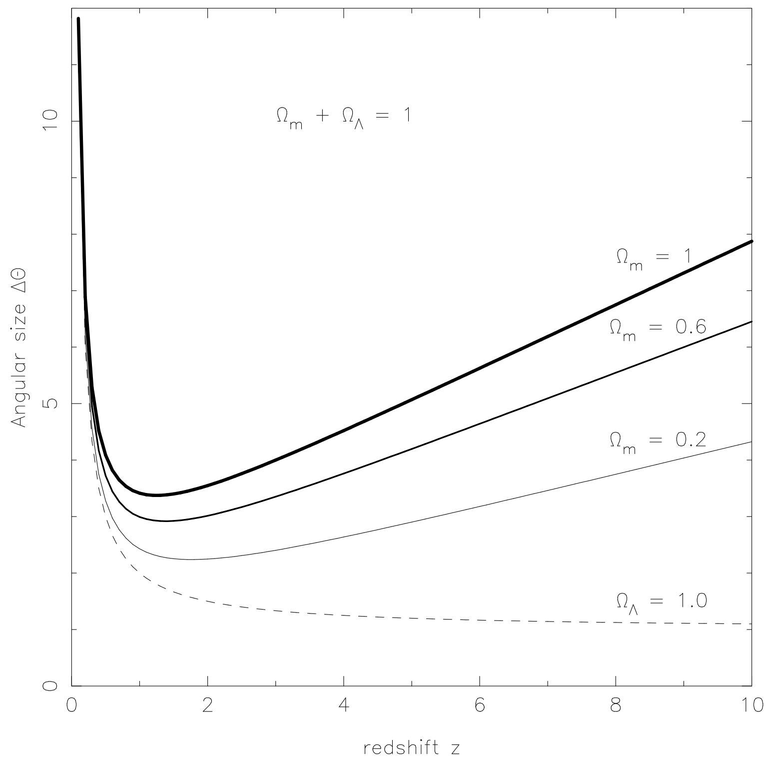

Fig. 10. The angular size is shown as a function of cosmological redshift z for flat cosmological models with a cosmological constant $\Omega_m + \Omega_\Lambda = 1$. Heavier lines correspond to larger values of Ω_m. For comparison we also show (dashed line) the angular size in a de Sitter universe ($\Omega_\Lambda = 1$).

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object at the location of the observer

$$\Delta\theta = \frac{D}{d_A} \tag{47}$$

where $d_A = a(t)r$ is the 'angular-size distance'. Since $1 + z = a_0/a(t)$ one gets $d_A = d_L(1+z)^{-2}$, where $d_L(z) = a_0 r(1+z)$ is the luminosity distance discussed in the previous section. Accordingly (47) may be rewritten as

$$\Delta\theta = \frac{D(1+z)^2}{d_L(z)}. \tag{48}$$

In Fig 10 we have plotted the angular size - redshift relation for flat cosmological models with a cosmological constant. (We have used expressions (28) for the luminosity distance d_L and (19) for the dimensionless Hubble parameter $h(z)$, assuming a flat universe $\Omega_{total} = \Omega_m + \Omega_\Lambda = 1$.)

We find that as the object is moved to higher redshifts its angular size first decreases (as naively expected) but soon begins to increase after passing through a minimum value. The appearance of a minimum angular size at a given redshift $z_{\min}$ is a generic feature of cosmological models with $\Omega_m > 0$. Differentiating (48) with respect to the redshift after substituting for d_L from (30), and then setting $\delta\Delta\theta/\delta z = 0$ gives $z_{\min} = 1.25$ for the flat matter dominated Einstein-de Sitter universe. ¿From figure 10 we find that the location of the minimum angular size moves to higher redshifts as Ω_Λ is increased, until in the limiting case $\Omega_\Lambda = 1$ there is no minimum at all. (Formally $z_{\min} \to \infty$ in de Sitter space, indicating that the angular size of an object decreases monotonically with redshift without ever reaching a minimum value.)

The suggestion that angular sizes of galaxies could be used to discriminate between cosmological models was first made in [95]. Curiously the angular size of a typical galaxy at a redshift $z \sim 1$ is roughly one arc second which is close to the limiting value of the angular resolution ('seeing') allowed by the Earth's atmosphere [154]. Beyond $z \sim 1$ the angular size of an object increases, and if one is confident that galaxies of a given class at higher redshifts are similar in form to their lower redshift counterparts, then this test can in principle provide a powerful means of discriminating between world models especially with the use of satellite data which can get around the 'seeing' limit. Other (larger) objects which can be used to probe the angular size-redshift relation include clusters of galaxies [142] and both extended and compact radio galaxies [105,31]. Extended radio sources which include the twin radio lobes surrounding a radio galaxy can have sizes ranging from a few kpc to ~ 1000 kpc, consequently the typical angular size of such objects is ~ 20 arc seconds which can easily be measured using ground based techniques.

However, a word of caution must now be added, both clusters and radio galaxies are prone to strong evolutionary effects which could lead to a change in size over cosmological epoch. Thus a comprehensive understanding of physical effects associated with both clusters (subclustering, virialization etc.) and radio galaxies (evolution of radio lobes and the central engine etc.) is necessary before the angular size-redshift relation can be used to unambiguously determine cosmological parameters including Ω_Λ.

Recently Kellerman (1993) and Gurvits et al. (1998) have studied the angular sizes of compact radio sources (QSO's and AGN's) arguing that the central 'engine' powering these objects is likely to be controlled by a limited number of physical parameters (mass of central black hole, accretion rate etc.) and may therefore be subject to less evolutionary effects than extended radio sources. On the basis of an analysis of a large number of sources spanning a wide redshift range $0.01 < z < 4.73$ these authors claim that an increase in the angular size has been detected which is consistent with $\Omega_m \sim 1$. (However working with the same data set as Kellerman (1993), Kayser (1995) has shown that a significant Λ cannot be ruled out.)

An interesting feature of closed universes not present in the flat models considered in figure 10, or in open models, is the presence of antipodal points. The presence of antipodes can contribute to changing the angular size as well as to the lensing of a source galaxy or quasar [80] and therefore provide us with a good means to constrain closed cosmological models. A metric describing the closed FRW universe is

$$ds^2 = c^2 dt^2 - a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] \tag{49}$$

where $0 \le \chi, \theta \le \pi, 0 \le \phi \le 2\pi$. If we assume that the observer is located at $\chi = 0$ then the associated antipodal point is at $\chi_a = \pi$. Substituting $ds^2 = 0$ we obtain $\chi = \eta \equiv \int dt/a$, where η is the conformal time coordinate. In a matter dominated universe the form for the expansion factor is $a(\eta) = A(1 - \cos\eta)$, $ct = A(\eta - \sin\eta)$, and the Hubble parameter is given by $H \propto \sin\eta/(1 - \cos\eta)$ where $0 \le \eta \le 2\pi$. Thus a light ray from the antipodal point $\chi_a = \pi$ reaches $\chi = 0$ at the time of maximum expansion $\eta = \pi$ (corresponding to $H = 0$). Consequently in a matter dominated closed universe, light from an antipodal point can *never* reach an observer during the expanding phase (when $H > 0$). This situation changes when one considers a closed universe with a cosmological term. In this case the universe is not obliged to recollapse and one can observe an antipodal point during the expansion epoch. The location of antipodal points can be derived from the following considerations: from (28) we find for a closed universe

$$d_L(z) = \frac{(1+z)H_0^{-1} Sin(\eta_0 - \eta)}{|\Omega_{total} - 1|^{\frac{1}{2}}} \tag{50}$$

where

$$\eta_0 - \eta = |\Omega_{total} - 1|^{\frac{1}{2}} \int_0^z \frac{dz'}{h(z')} \tag{51}$$

since $d_A = d_L(1+z)^{-2}$ it follows that for

$$\eta_0 - \eta = n\pi, \quad d_A \simeq 0 \tag{52}$$

and, from (47), $\Delta\theta \to \infty$, *i.e.* the angular size of an object located close to one of the antipodal points (52) can become very large. Consequently such an object will appear to us to be extremely bright even if located at a high redshift ! The presence of 'normal' galaxies and quasars, as well as gravitationally lensed objects out to redshifts $\simeq 4.92$ set a lower limit on the antipodal redshift $z_a(\Omega_\Lambda, \Omega_m) > 4.92$ which can be used to constrain the cosmological parameter pair $(\Omega_\Lambda, \Omega_m)$ in a closed universe [51,124,147]. (Multiple images of a source object located further away than the antipodal redshift z_a are very difficult to form [80].) Since the supernovae analysis prima-facie appears to favour a closed universe, antipodal constraints may be used to further narrow down the allowed range in parameter space.

Observations of large scale structure indicate that the model which comes closest to explaining most observational features of galaxy clustering is ΛCDM, a model containing a cosmological constant in addition to baryons and cold dark matter [114,144]. Parameters of this model which agree well with observations are $\Omega_\Lambda \mathrm{h}^2 \simeq 0.33$, $\Omega_b \simeq 0.02$, $\Omega_m \simeq 0.3$, where $\mathrm{h} = H_0/100$ is the Hubble parameter in units of 100 km/sec/Mpc. (Setting $h = 0.7$ gives $\Omega_\Lambda = 0.68$.)

There are several reasons as to why the presence of Λ improves the performance of the standard cold dark matter model. The first is related to the fact that in a spatially flat universe linearized density perturbations grow at a slower rate in the presence of Λ than in its absence. (The growth rate is however faster than that in an open universe for identical values of $1 - \Omega_m$.) This changes the initial normalization of the density field since the linearized gravitational potential now becomes time-dependent, which affects the Sachs-Wolfe integral discussed in section 4.4. The slow down in the rate of growth also affects the abundance of very massive objects (clusters and superclusters) some of which may have formed only relatively recently and would therefore feel the presence of long wavelength modes still in the linear regime. A small value of Ω_m (alternatively, a large value of $\Omega_\Lambda = 1 - \Omega_m$) also affects the matter power spectrum in ΛCDM models which is strongly influenced by the epoch of matter radiation equality. This effect is incorporated in the *shape parameter* [9] $\Gamma = \Omega_m h$: a small value of Ω_m leads to a larger value of the horizon at matter-radiation equality $d_{eq} \simeq 16/(\Gamma h)$ Mpc and hence to more long wavelength power in the fluctuation spectrum $P(k) = \langle|\delta_k|^2\rangle$. Both open models and ΛCDM models show better agreement with galaxy clustering data on large scales [54], the 'best fit' value of Γ being $\Gamma \simeq 0.25$.

An independent estimate of Ω_m is provided by the peculiar velocities of galaxies in our neighborhood (on scales $\sim 10 - 100\ \mathrm{h}^{-1}$ Mpc). The results of a joint estimate from velocity flows and supernovae gives the most likely values $\Omega_m \simeq 0.5$ and $\Omega_\Lambda \simeq 0.8$, thereby favouring an approximately flat universe [45].

A low value of Ω_m is also indicated by studies of clusters of galaxies. Clusters of galaxies have traditionally been powerful probes of cosmological structure formation scenario's. The masses of rich clusters can be estimated using three independent methods: the velocity dispersion of member galaxies, the cluster X-ray temperature due to hot intracluster gas and strong gravitational lensing of background galaxies by the cluster. All three methods provide an estimate of the cluster mass which ranges from 10^{14} to $10^{15}\ \mathrm{h}^{-1}\ \mathrm{M}_\odot$ for the mass located

[9] The shape parameter is so named because it affects the shape of the Power spectrum $P(k)$, which interpolates between the asymptotic regimes [170] $P(k) \propto k$ for $k \to 0$ and $P(k) \propto k^{-3}\log^2 k$ for $k \to \infty$. The maximum value of $P(k)$ occurs near $k \sim d_{eq}^{-1}$.

within the central 1.5h^{-1} Mpc. region of a cluster [6]. The resulting median mass-to-light ratio for rich clusters is $M/L_B \simeq 300 \pm 100\text{h}\ M_\odot/L_\odot$, which when integrated over the full range of luminous matter in the universe gives an estimate for the density parameter $\Omega_m = 0.2 \pm 0.1$.

A low value of Ω_m is also indicated by a study of baryonic matter within clusters. In a detailed study of the composition of the Coma cluster which included estimates of the baryonic mass fraction provided by X-ray emitting gas and virial measurements of its total mass, White et al (1993) showed that the ratio of baryonic matter to total mass $\Omega_b h^{3/2}/\Omega_m = 0.07 \pm 0.03$. As a result the baryonic mass fraction greatly exceeds nucleosynthesis constraints $\Omega_b h^2 = 0.015 \pm 0.005$ if $\Omega_m = 1$, leading to a 'baryon catastrophe'. However no catastrophe occurs if $\Omega_m h^{1/2} = 0.21 \pm 0.12$ since the value of Ω_b is now small enough to be acceptable by nucleosynthesis constraints [144]. This result therefore is strongly supportive of either an open universe or one that is Λ dominated and flat, so that $\Omega_m = 1 - \Omega_\Lambda \ll 1$.

Observations of cluster abundances can be used to provide good estimates of σ_8 – the average root-mean-square mass fluctuation in a sphere of radius 8h^{-1} Mpc. The best-fit value of σ_8 consistent with present day cluster abundances is $\sigma_8 \simeq 0.5\ \Omega_m^{-0.5}$. This value gives a measure of the clustering amplitude on small scales and therefore can be used to normalize the power spectrum of density perturbations. A complementary method of normalization is provided by large angle CMB anisotropies measured by COBE. Taken together the σ_8 normalization on small scales and the COBE normalization on large scales (~ 1000 Mpc.) provide very useful constraints on the cosmological parameters $\Omega_m, \Omega_\Lambda, \Omega_B$, on the biasing parameter $b = \delta_{lum}/\delta_{dark}$ and on the 'primordial tilt' in the power spectrum $|\delta_k|^2 \propto k^n$ which can be shown to lie in the range $|1-n| \lesssim 0.2$ [144].

A potentially powerful method for discriminating between different cosmological models is provided by the abundance of rich clusters of galaxies measured at high redshifts. The presence of large amounts of X-ray emitting gas in many rich clusters provides us with a useful observational tool with which to probe cluster mass. Observations of galaxy clusters are then matched against theoretical models which model cluster formation and evolution using either Press-Schechter techniques or N-body/Hydro-simulations [145,88,61,19,41,7,58,194]. As discussed earlier the growth of long wavelength perturbations which are still in the linear regime, is significantly slower in low density models (both with and without a cosmological constant) than in a critical density $\Omega_m = 1$ universe. This leads to dramatic differences in the redshift dependence of the rich cluster abundance in cosmological models: rich clusters are much rarer at high redshifts in an $\Omega_m = 1$ universe than they are in a low density universe (see figure 11). For instance, whereas almost all massive clusters with $M \sim 10^{15} M_\odot$ are expected to have formed by $z \sim 0.5$ in a low density uni-

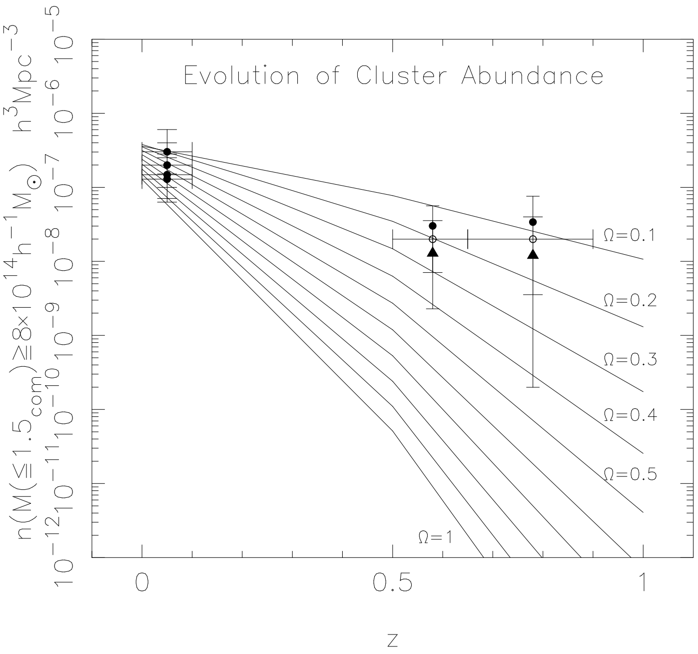

Fig. 11. The observed and expected cluster abundance is shown as a function of redshift for massive clusters with $M_{cl} \gtrsim 8 \times 10^{14} M_\odot$ located within the Abell radius of $1.5\mathrm{h}^{-1}$Mpc. The curves show the expected cluster abundances in CDM models with different Ω_m. Figure courtesy of Neta Bahcall (1999).

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verse, only a small fraction ($< 10\%$) of the present day $10^{15} M_\odot$ clusters would have been in place by $z \sim 0.5$ in an $\Omega_m = 1$ universe [78,194]. The existence of three massive clusters in the redshift range $z \sim 0.5 - 0.9$ has therefore been viewed as a difficulty for the standard cold dark matter model with $\Omega_m = 1$ for which 10^{-3} rich clusters are expected at $z > 0.5$ [78,7,6]. It must be noted however that large uncertainties in both the observational data (only a few very massive clusters have been reliably observed at high z) and in our theoretical understanding of rich clusters, makes it difficult at present to place unambiguous constraints on the values of Ω_m and Ω_Λ [195]. It is hoped that better quality data from satellite launches planned for the immediate future (XMM) and more accurate modelling of large scale structure will improve the situation significantly in the near future.

Constraints on the abundance of rich clusters also come from *arcs* caused by the strong gravitational lensing of extended background sources (galaxies, radio sources) by foreground clusters. Since clusters act as gravitational lenses for background sources, the larger number of clusters at early epochs in (i) open, low Ω_m models, and (ii) flat, high Ω_Λ models, relative to (iii) flat $\Omega_m = 1$ models leads to a greater abundance of arcs in both (i) and (ii) relative to (iii). An estimate by Bartelmann et al. (1998) based on numerical simulations of large scale structure, has shown that an order of magnitude more arcs are predicted in flat models with $\Omega_m \simeq 0.3$, $\Omega_\Lambda \simeq 0.7$ ($\mathcal{N}_{\rm arcs} \sim 280$) than in the flat $\Omega_m = 1$ model ($\mathcal{N}_{\rm arcs} \sim 36$). In open models this effect is even more dramatic ($\mathcal{N}_{\rm arcs} \sim 2400$ for $\Omega_m \simeq 0.3, \Omega_\Lambda = 0$). However, impressive as these results are, the absence of a comprehensive data base for arcs and uncertainties in the modelling of galaxy clusters makes it difficult to attempt to constrain theoretical models on the basis of observations at present. (Bartelmann et al. (1998) however make a case for a low density universe by arguing that the observed number of arcs in the EMSS arc survey extrapolated to the full sky is 1500 - 2300, which is close to what one observes for low density models in their numerical simulations.) Both observational data sets and the theoretical modelling of clusters are likely to improve significantly in the near future giving this method potentially great importance in the ongoing 'quest for Λ'.

Finally, the Lyman-α forest which populates the spectra of quasars provides a potentially powerful means of discriminating between rival models of structure formation and in probing the presence of a cosmological Λ-term at intermediate redshifts $0 \leq z \leq 5$ [100,206].

5. Theoretical issues: Vacuum fluctuations and the Cosmological constant

Having summarised the observational evidence for a cosmological Λ-term let us now turn our attention to some theoretical implicationsi of Λ.

A turning point in our understanding of the cosmological constant occurred when Zeldovich (1968), intrigued by Λ-based cosmological models presented to explain an excess of quasars near redshift ~ 2, showed that zero-point vacuum fluctuations must have a Lorentz invariant form $P_{vac} = -\rho_{vac} c^2$, equivalently $T_{ik}^{vac} = \Lambda g_{ik}$, *i.e.* the vacuum within the quantum framework had properties identical to those of a cosmological constant.

Let us review this situation beginning with an oscillator consisting of a single particle of mass m moving under the influence of a potential $V = \frac{1}{2}kx^2$. At the classical level one expects the lowest energy state to be associated with the particle at rest at $x = 0$, so that the total energy vanishes: $E = T + V = 0$. Thus, within the classical framework, the vacuum can be viewed as a state having zero energy and momentum. However when viewed in terms of quantum mechanics the situation changes, the uncertainty relation preventing the particle (wave function) from simultaneously having a fixed location ($x = 0$) and a fixed velocity ($T = 0$). As a result, the ground state energy of the oscillator is finite and is given by $E = \frac{1}{2}\hbar\omega$, where $\omega = k/m$. Turning now to quantum theory, it is well known that after secondary quantization a classical field can be looked upon as an ensemble of oscillators each with frequency $\omega(k)$. The net 'zero-point energy' of this field is $E = \sum_k \frac{1}{2}\hbar\omega(k)$. Thus the uncertainty relation endows the vacuum with both energy and pressure!

The existence of zero-point vacuum fluctuations has been spectacularly demonstrated by the Casimir effect. [10] The vacuum energy associated with zero-point fluctuations is formally infinite and results in a 'cosmological constant problem' for the universe [205]. Because of the importance of this result we shall perform a simple calculation aimed at evaluating the zero-point energy associated with a quantized scalar field in flat space-time. (The reader is referred to Birrell & Davies 1982 for a discussion of quantization of higher spin fields.)

Consider the action defined in flat four dimensional space-time [11]

$$\mathcal{S} = \int \mathcal{L}(x) d^4x \tag{53}$$

where $\mathcal{L}(x)$ is the Lagrangian density for a massive scalar field

$$\mathcal{L} = \frac{1}{2}(\eta^{ij}\Phi_{,i}\Phi_{,j} - m^2\Phi^2) \tag{54}$$

propagating in flat space-time with metric η_{ij}.

The variational principle $\delta_\Phi \mathcal{S} = 0$ gives the Klein-Gordon equation

$$(\Box + m^2)\Phi = 0 \tag{55}$$

where $\Box \equiv \eta^{ik}\partial_i\partial_k$.

To quantize the system we treat the field Φ as an operator

$$\Phi(x) = \sum_{\mathbf{k}} [a_k\phi_k(\mathbf{x}, \eta) + a_k^\dagger\phi_k^*(\mathbf{x}, \eta)] \tag{56}$$

where $a_k,\ a_k^\dagger$ are annihilation and creation operators $[a_k, a_{k'}^\dagger] = \delta_{kk'}$, defining the vacuum state $a_k|0\rangle = 0,\ \forall k$. An orthonormal set of solutions defined using periodic boundary conditions on a three dimensional torus of side L is given

[10] The Casimir effect arises because vacuum fluctuations satisfy the quantum mechanical wave equation and hence are sensitive to boundary conditions. As shown by Casimir (1948) the presence of two flat parallel conducting plates at a separation l, alters the distribution of electromagnetic field modes existing in the vacuum, resulting in an attractive force per unit area between the plates: $F = -\hbar c\pi^2/240l^4$ which is of vacuum origin. The Casimir effect has been experimentally measured by Spaarnay (1957) and others [18,189,205,139].

[11] Following Landau & Lifshitz (1975) we use Latin indices to describe space-time coordinates, so that i,j,k... = 0,1,2,3; and Greek indices to describe spatial coordinates: α, β, γ.. = 1,2,3.

by [18]

$$\begin{aligned}\phi_{\mathbf{k}} &= \frac{1}{\sqrt{2L^3\omega}}\exp\left(i\mathbf{k}\mathbf{x} - i\omega_k t\right)\\ k_j &= \frac{2\pi n_j}{L}, \quad n_j \in I\end{aligned} \tag{57}$$

where $\omega_k^2 = k^2 + m^2$, and the field modes have been normalized using

$$(\phi_{\mathbf{k}}, \phi_{\mathbf{k}'}) = \delta_{\mathbf{k}\mathbf{k}'} \tag{58}$$

where

$$(\phi_1, \phi_2) = -i\int [\phi_1\partial_t\phi_2^* - \phi_2^*\partial_t\phi_1]d^3x. \tag{59}$$

Consider next, the energy-momentum tensor

$$T_{ij} = \Phi_{,i}\Phi_{,j} - \frac{1}{2}\eta_{ij}\eta^{kl}\Phi_{,k}\Phi_{,l} + \frac{1}{2}m^2\Phi^2\eta_{ij} \tag{60}$$

where T_{00} defines the energy density

$$T_{00} = \frac{1}{2}(\dot{\Phi}^2 + \partial_\mu\phi\partial^\mu\Phi + m^2\Phi^2) \tag{61}$$

and $T_{0\alpha}$ the momentum density

$$T_{0\alpha} = \frac{\partial\Phi}{\partial t}\frac{\partial\Phi}{\partial x^\alpha}, \quad \alpha = 1, 2, 3. \tag{62}$$

Substituting from (56) & (57) into (60) one obtains for the Hamiltonian H

$$H \equiv \int T_{00}d^3x = \frac{1}{2}\sum_{\mathbf{k}}(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}}a_{\mathbf{k}}^\dagger)\omega_{\mathbf{k}} \tag{63}$$

which can be further simplified using the commutation relation $[a_k, a_{k'}^\dagger] = \delta_{kk'}$ to

$$H = \sum_{\mathbf{k}}(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2})\omega_{\mathbf{k}}. \tag{64}$$

A similar operation on the momentum density yields [18]

$$P_\alpha \equiv \int T_{0\alpha}d^3x = \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}k_\alpha, \quad \alpha = 1, 2, 3. \tag{65}$$

Inspecting expressions (63) and (65) for the Hamiltonian H and the momentum operator P_α we find, for the expectation value of these quantities in the vacuum state $|0\rangle$

$$\langle 0|\mathbf{P}|0\rangle = 0, \qquad\qquad \langle 0|H|0\rangle = \frac{1}{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}. \tag{66}$$

Transforming the sum $\sum_k$ to an integral we get

$$\frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}} = \frac{1}{2}(\frac{L}{2\pi})^3 \int \omega(\mathbf{k}) d^3k = \frac{L^3}{4\pi^2}\int_0^\infty \sqrt{k^2+m^2}\; k^2 dk. \tag{67}$$

From (66) & (67) we find that the energy density of zero-point vacuum fluctuations is dominated by ultraviolet divergences which diverge as k^4 when $k \to \infty$. The vacuum state therefore has zero momentum and infinite energy ! (In terms of Feynman diagrams the energy density of zero-point fluctuations is associated with a one-loop vacuum graph, see figure (13).)

Within the framework of Newtonian gravity and either classical or quantum mechanics, an infinite (or very large) vacuum energy does not cause serious problems since interaction between particles is governed not by the absolute value of the potential energy V, but by its gradient ∇V. As a result one can always redefine $V' \to V + V_0$ so that the minimum of V' has zero net energy. The situation changes dramatically when we view the vacuum within the framework of general relativity. A central tenet of the general theory of relativity is that the gravitational force couples to *all* forms of energy through the Einstein equations $G_{ik} = \frac{8\pi G}{c^4}T_{ik}$. Therefore if the vacuum has energy then it also gravitates ! In order to probe this effect further one needs to know the equation of state possessed by the vacuum energy, equivalently the form of its energy momentum tensor $T^{ik}_{\rm vac}$. This question was answered by Zeldovich (1968) who showed that the vacuum state had to have a Lorentz-invariant form, one that was left unchanged by a velocity transformation and hence appeared the same to all observers. This requirement is exactly satisfied by the equation of state $P = -\rho$ possessed by the cosmological constant, since the relation $T_{ik} = \Lambda g_{ik}$ is manifestly Lorentz-invariant. [12]

All fields occurring in nature contribute an energy density to the vacuum and expressions analogous to (66) for bosons can also be derived for fermions. Since

[12] Zero-point fluctuation s are usually regularized by 'normal ordering' – a rather ad hoc procedure which involves the substitution $a_k a_k^\dagger \to a_k^\dagger a_k$ in (63). In curved space-time a single regularization is not enough to rid $\langle T_{ik}\rangle$ of all its divergences. Three remaining 'infinities' must be regularized, leading to the renormalization of additional terms in the one-loop effective Lagrangian for the gravitational field, which, in an FRW universe becomes: $\mathcal{L}_{\rm eff} = \sqrt{-g}[\Lambda_\infty + R/16\pi G_\infty + \alpha_\infty R^2 + \beta_\infty R_{ij}R^{ij}]$. Renormalization of the first term $\Lambda_\infty \to 0$ corresponds to normal ordering. The presence of the second term $R/16\pi G_\infty$, led Sakharo v to postulate that the gravitational field might be 'induced' by one-loop quantum effects in a curved background geometry, since one could recover the ordinary Einstein action by renormalizing the 'bare' value G_∞ to its observed value: $G_\infty \to G_{\rm obs}$ [173]. Thus *both* the cosmological constant Λ and the gravitational constant G may be induced by quantum effects. The remaining two terms in $\mathcal{L}_{\rm eff}$ give rise to vacuum polarization effects and have been extensively discussed in the literature [18,81].

fermionic creation, annihilation operators anti-commute this leads to

$$\langle 0|H_f|0\rangle = -\frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}. \tag{68}$$

Comparing (68) with (66) we find that the zero-point energy of fermions is equal and opposite to that of bosons (having identical mass).

The advent of Supersymmetry in the 1980s, incorporating a fundamental symmetry between bosons and fermions, led to the hope that the cosmological constant problem would finally be resolved, since the one-to-one correspondence between bosons and fermions in such theories was expected to lead to cancellation between bosonic and fermionic infinities [216]. However Supersymmetry is expected to exist only at very high energies/temperatures. At low temperatures such as those existing in the universe today, Supersymmetry is broken. One might therefore expect the cosmological constant to vanish in the early universe only to reappear later, when the universe has cooled sufficiently so that $T \ll T_{SUSY} > 10^3$ GeV. Thus the cosmological constant problem re-emerges to haunt the present epoch !

Although the cosmological constant problem remains unresolved, an important aspect of Zeldovich's work was that it demonstrated a firm physical mechanism for the generation of a cosmological constant. Later work, mostly associated with Inflationary model-building, further strengthened this idea by showing that an effective cosmological constant could arise due to diverse physical processes including symmetry breaking, vacuum polarization in curved space-time, higher dimensional 'Kaluza-Klein' theories etc. Some of these developments have been reviewed in [132].

6. The cosmological constant and spontaneous symmetry breaking

An important development in our understanding of the 'vacuum energy' was associated with the phenomenon of symmetry breaking in the electroweak Weinberg-Salam model. Consider the scalar field action

$$S = \int \sqrt{-g}\mathcal{L}d^4x \tag{69}$$

where $\mathcal{L}$ is the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\, g^{ij}\partial_i\phi\partial_j\phi - V(\phi) \tag{70}$$

and the scalar field potential has the form

$$V(\phi) = V_0 - \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4. \tag{71}$$

This particular form of the potential (illustrated in Fig. 12) endows the system with some interesting properties. For instance since the symmetric state $\phi = 0$ is unstable ($V''(\phi) < 0$) the system settles in the ground state $\phi = +\sigma$ or $\phi = -\sigma$, where $\sigma = \sqrt{\mu^2/\lambda}$ thus breaking the reflection symmetry $\phi \leftrightarrow -\phi$ present in the Lagrangian. The energy momentum tensor T_{ik} of a scalar field with Lagrangian density $\mathcal{L}$ is given by

$$T_{ik} = \phi_{,i}\phi_{,k} - g_{ik}\mathcal{L}. \tag{72}$$

Assuming ϕ to be homogeneous and time-independent one finds the ground state energy-momentum tensor to be

$$T_{ik} = g_{ik}V(\phi = \sigma), \tag{73}$$

the vacuum state therefore has precisely the form of an effective cosmological constant $T_{ik} = g_{ik}\Lambda_{eff}$ where $\Lambda_{eff} = V(\phi = \sigma) = V_0 - \mu^4/4\lambda$. Setting $V_0 = 0$, results in a *negative* cosmological term $\Lambda_{eff} = -\mu^4/4\lambda$. Substituting parameters arising in the electroweak theory results in a lower limit on the value of the vacuum energy density [205] $\rho_{vac} = |\Lambda_{eff}|/8\pi G = 10^6\mathrm{GeV}^4$, which is almost 10^{53} times larger than current observational upper limits on the cosmological constant $\rho_{vac,0} = \Lambda_0/8\pi G \sim 10^{-29}\mathrm{g/cm}^3 \simeq 10^{-47}\mathrm{GeV}^4$. Clearly in order not to violate observational bounds today, one must set $V_0 \simeq \mu^4/4\lambda$ so that $\Lambda_{eff} \sim \Lambda_0$. An interesting feature of this 'regularization' of the cosmological constant is that, while drastically reducing the value of the cosmological constant today it simultaneously generates a large cosmological constant $\sim V_0$ during an early epoch before symmetry breaking, thereby giving rise to the possibility of Inflation ! The cosmological constant problem therefore presents us with a dilemma: it is certainly good to have a large cosmological constant during an early epoch so as to resolve – via Inflation – the horizon and flatness problems and possibly generate seed fluctuations for galaxy formation. However one must simultaneously ensure that the value of Λ today is small so as not to conflict with observations. As we have seen, in models with SSB this dual requirement of '*large Λ in the past + small Λ at present*' results in an enormous fine tuning of initial conditions.

7. Mechanisms for generating a small current value of Λ.

As we saw in the last two sections, the vacuum associated with both one-loop quantum effects and models with spontaneous symmetry breaking, has

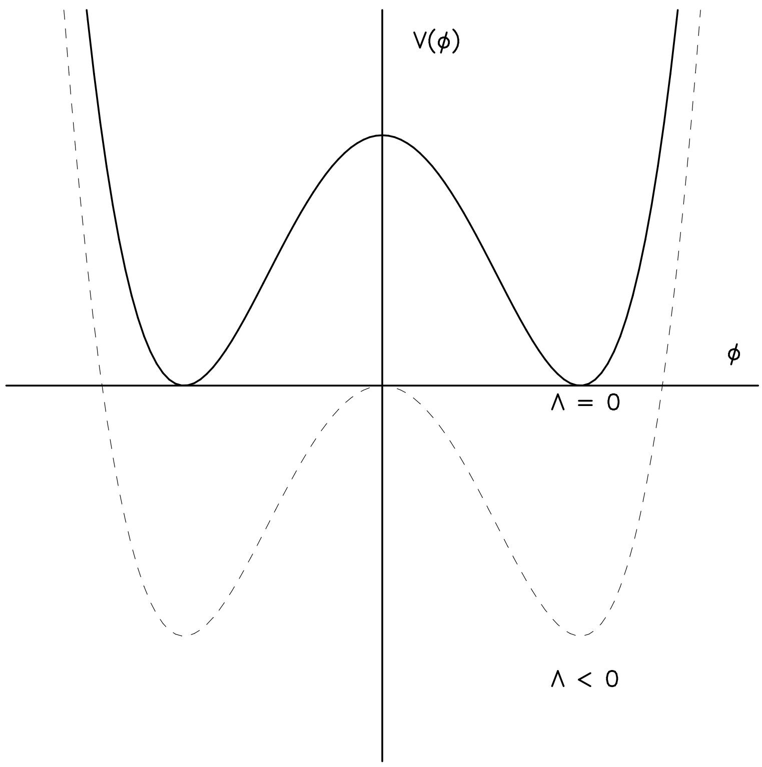

Fig. 12. The 'Mexican top-hat' potential describing spontaneous symmetry breaking shown: before (dashed) and after (solid) the cosmological constant has been 'renormalized'.

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properties identical to those of a cosmological constant. There is one problem however, in the case of zero-point fluctuations, the vacuum density turns out to be infinite leading to an infinitely large cosmological term and resulting in a *cosmological constant problem* for cosmology (see section 5). Assuming that the ultraviolet divergences responsible for the cosmological constant problem can be cured by (hitherto unknown) physics occurring near the Planck scale, one gets a finite but very large value

$$\rho_\Lambda = \Lambda c^2/8\pi G \simeq \rho_{Pl} = c^5/G^2\hbar \sim 5 \times 10^{93}\ \mathrm{g\ cm^{-3}},$$

where ρ_{Pl} is the Planck density. On the other hand, as we saw earlier, recent observations of the luminosities of high redshift supernovae combined with CMB results give the following value for the dimensionless density in Λ

$$\Omega_\Lambda = \rho_\Lambda/\rho_{cr} \equiv \frac{\Lambda c^2}{3H_0^2} \simeq 0.7$$

where $\rho_{cr} = 3H^2/8\pi G = 1.88 \times 10^{-29}\mathrm{h}^2\ \mathrm{g/cm^3}$ (see sections 4.3 & 4.4), which leads to $\rho_\Lambda \simeq \rho_{Pl} \times 10^{-123}$, *i.e.* the value of the cosmological constant today is almost 123 orders of magnitude smaller than the Planck density !

As we have shown in section 6, a large (negative) value of the vacuum energy also arises in models with spontaneous symmetry breaking. In this case, the fine tuning involved in matching the present value of Λ to observations depends upon the symmetry breaking scale, and ranges from 1 part in 10^{123} for the Planck scale, to 1 part in 10^{53} for the electroweak scale.

Clearly the question begging an answer is: which physical processes can generate a small value for Λ today without necessarily involving a delicate fine tuning of initial conditions? Although no clear cut answers are available at the time of writing (it may even be that a very small Λ may demand completely new physics) some avenues which could lead us to interesting answers will be explored in this section.

7.1. A Decaying Cosmological Constant?

One method of resolving the dilemma between a very large cosmological constant (predicted by field theory) and an extremely small one (suggested by observations) with obvious cosmological advantages is to make the cosmological term time-dependent. An initially large cosmological term would give rise to Inflation, ameliorating the horizon and flatness problems and (possibly) seeding galaxy formation. The subsequent slow decay of $\Lambda(t)$ would enable a small present value $\Lambda(t_0)$ to be reconciled with observations suggesting $\Omega_\Lambda \sim 0.7$. (A time dependent cosmological term of course arises in Inflationary models and during cosmological phase transitions, but in such cases the post-inflationary decay of the cosmological term is very rapid.)

The first proposal for dynamically reducing the cosmological constant was made by Dolgov (1983) who considered a massless non-minimally coupled scalar field having the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\phi^{,l}\phi_{,l} - \xi R\phi^2) \tag{74}$$

and the resulting equation of motion

$$\Box\phi + \xi R\phi = 0, \tag{75}$$

where R is the scalar curvature and ξ the coupling to gravity. Considering the special case of a homogeneous scalar field, the Einstein equations become

$$3H^2 = \Lambda + 8\pi G(\rho_\phi + \rho_{matter}) \tag{76}$$

where

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + 3\xi H^2\phi^2 + 6\xi H\phi\dot{\phi} \tag{77}$$

is the scalar field energy density. The scalar field equation (75) reduces to

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + 6\xi\left[\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2\right]\phi = 0. \tag{78}$$

Dolgov made the discovery that, for negative values of ξ, the scalar field is unstable: its energy density ρ_ϕ becomes large and negative compensating for the cosmological constant in (76), so that the resulting *effective* cosmological constant rapidly decays to zero. Let us demonstrate this by examining the Einstein equation (76) which together with the scalar field equation (78) defines a pair of nonlinear differential equations determining the behaviour of the scale factor $a(t)$ and the scalar field $\phi(t)$. The term $3\xi H^2\phi^2$ in ρ_ϕ can be carried over into the left hand side of (76) resulting in

$$3H^2 \simeq \frac{\Lambda}{1 - 8\pi G\xi\phi^2} + ... \tag{79}$$

As Dolgov demonstrated, $\phi(t)$ grows with time if $\xi < 0$, so that the effective cosmological constant $\Lambda_{eff} = \Lambda/(1 + 8\pi G|\xi|\phi^2)$ decreases. The late time behaviour of $a(t)$, $\phi(t)$ obtained by solving (76 - 78) with $\rho_m \ll \rho_\phi$ has the asymptotic form [46,66]

$$a \propto t^q, \ q = \frac{1}{2} + \frac{1}{4|\xi|}, \ \phi \propto t.$$

As a result, $\lim_{t\to\infty}\Lambda_{eff} \to 0$ *i.e.* the cosmological term vanishes at late times.

Unfortunately this mechanism cannot be used in real universe. The first problem with this approach is that the very mechanism which decreases the cosmological constant also quenches the effective gravitational constant, since from (76),

$$G_{eff} = \frac{G}{1 - 8\pi G\xi\phi^2} \to 0 \quad \text{as} \quad t \to \infty. \tag{80}$$

As a result, the effective gravitational constant becomes noticeably time-dependent: $\dot{G}_{eff}/G_{eff} = -2/t \sim -10^{-10}$ yr^{-1}, which strongly contradicts upper limits from Viking radar ranging [87] and lunar laser ranging experiments [211]. Another problem is that such screening of Λ is still not sufficient. The remaining part of Λ remains of the order of the Ricci tensor all the time, while we need it to be much less than the Ricci tensor during the matter dominated epoch to obtain sufficient growth of scalar perturbations. Finally, $\Omega_m \ll 1$ during this regime.

An extension of this method to higher spin fields (massless vector and tensor) can remove the first drawback by making a cancellation of the cosmological constant possible while keeping the gravitational coupling constant time-independent [47]. However, the other difficulties (especially, the second one) remain. This shows that it is not easy to explain the observed Λ-term by a cancellation mechanism. Still this aesthetically attactive possibility should be investigated further (some variants of the early Dolgov mechanism are discussed in [10,205]).

7.2. Vacuum polarization and the value of Λ

Zeldovich (1968), having demonstrated that the energy density of the vacuum was infinite at the one-loop level, suggested that after the removal of divergences, the 'regularized' vacuum polarization contributed by a fundamental particle of mass m would be described by the expression

$$\rho_\Lambda \sim \frac{Gm^2}{\hbar c} m \; (\frac{mc}{\hbar})^3. \tag{81}$$

One can arrive at this result by means of the following argument: the vacuum consists of virtual particle-antiparticle pairs of mass m and separation $\lambda = \hbar/mc$. Although the regularized self-energy of these pairs is zero, their gravitational interaction is finite and results in the vacuum energy density $\epsilon_{vac} \equiv \rho_{vac} c^2 \sim \frac{Gm^2}{\lambda}/\lambda^3 = Gm^6c^4/\hbar^4$ corresponding to (81). (In terms of Feynman diagrams this corresponds to the energy associated with the two-loop vacuum graph shown in figure 13.) Substituting $m \to m_e(m_p)$ we find that the electron (proton) mass gives too small (large) a value for ρ_Λ. On the other hand, the pion mass gives just the right value [106] [13]

$$\rho_\Lambda = \frac{1}{(2\pi)^4} \; \rho_P \; (\frac{m_\pi}{M_P})^6 \simeq 1.3 \times 10^{-123} \rho_P = 6.91 \times 10^{-30} \; g \, cm^{-3} \; . \tag{83}$$

Finally, a small value of Λ can be derived from dimensionless fundamental constants of nature using purely numerological arguments. For instance, the fine structure constant $\alpha \equiv e^2/\hbar c \simeq 1/137$ when combined with the Planck scale ρ_P, suggests the relation [187]

$$\rho_\Lambda = \frac{\rho_P}{(2\pi^2)^3} e^{-2/\alpha} \simeq 1.2 \times 10^{-123} \rho_P = 6.29 \times 10^{-30} \; g \, cm^{-3} \; . \tag{84}$$

Or, when expressed in terms of $\Omega_\Lambda = \frac{8\pi G \epsilon_\Lambda}{3H_0^2}$ we get $\Omega_\Lambda \mathrm{h}^2 = 0.335$, in excellent agreement with observations. In principle, α could be some other fundamen-

[13] The large difference between ρ_Λ obtained using (81) for the proton and its observed value prompted Zeldovich to suggest that Fermi's weak interaction constant G_F might play a role in determining the vacuum energy, so that

$$\rho_\Lambda \simeq \frac{G G_F m^8 c^5}{\hbar^7}. \tag{82}$$

Although this leads to some improvement, ρ_Λ for the proton is still several orders of magnitude larger than its observed value.

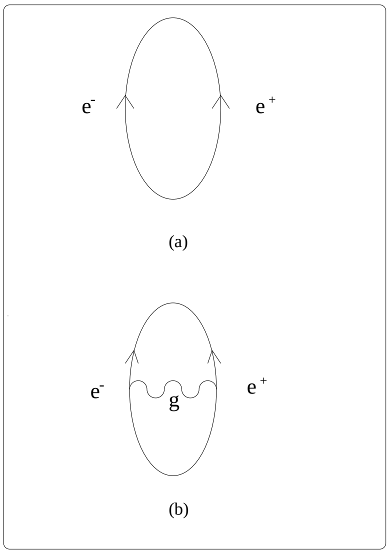

Fig. 13. This figure shows the one-loop (a) and two-loop (b) vacuum diagrams which contribute towards the vacuum energy density discussed in sections 5 & 7.2 respectively.

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tal constant, such as the 'string constant' associated with superstring theory, which might enter into exponentially small expressions for Λ of this type.

7.3. Late-time Inflation and Λ.

Conceivably, one might appeal to inflationary mechanisms which are so successful at generating a large cosmological constant during an early epoch to generate a small cosmological constant today. As pointed out in section 6, effective potentials giving rise to symmetry breaking generically predict a large negative value for a cosmological constant which has to be 'regularized' to give the small positive Λ observed today. The problem with these methods is that they usually prescribe an unevolving cosmological term whose present value is fixed at the time of symmetry breaking. This necessarily implies some fine tuning of parameters which can be as large as one part in 10^{123} (for symmetry breaking at the Planck scale) to one part in 10^{53} for the electroweak scale.

A different possibility is suggested by the family of potentials which lead to 'chaotic Inflation' $V \propto \phi^q$, $q \geq 2$. For instance $V = \frac{1}{2}m^2\phi^2$ will lead to the inflationary equation of state $P \simeq -\rho$ associated with a cosmological constant provided the scalar field rolls down its potential 'slowly' so that $\ddot{\phi} \simeq 0$ or

$(m/H_0)^2 \lesssim 1$. In other words, the Compton wavelength of the inflaton should be larger than the present Hubble radius $\lambda = \hbar/mc \gtrsim cH_0^{-1}$ suggesting an extremely small mass for the inflaton $m \lesssim 10^{-33}$ eV. One may be tempted to associate m with the small mass difference associated with solar neutrino oscillations $m = \Delta m_\nu^2/M_P \simeq 10^{-33}$ eV where $\Delta m_\nu^2 \simeq 10^{-5}$ eV^2, an idea which is speculative but not implausible [69,90].

7.4. Generating a small cosmological constant from Inflationary particle production.

A novel means of generating a small Λ at the present epoch was suggested by Sahni & Habib (1998).

Massive scalar fields in curved spacetime satisfy the wave equation

$$[\Box + \xi R + m^2]\Phi = 0 \tag{85}$$

where R is the Ricci scalar and ξ parametrizes the coupling to gravity. In a spatially flat FRW universe the field variables separate so that

$$\Phi_k = (2\pi)^{-3/2}\phi_k(\eta)\, e^{-i\mathbf{k}\cdot\mathbf{x}}$$

for each wave mode. The comoving wavenumber $k = 2\pi a/\lambda$ where λ is the physical wavelength of scalar field quanta. Defining the conformal field $\chi_k = a\phi_k$ and substituting $R = 6\ddot{a}/a^3$ into Eq. (85) leads to

$$\ddot{\chi}_k + [k^2 + m^2 a^2 - (1-6\xi)\ddot{a}/a]\chi_k = 0, \tag{86}$$

where differentiation is carried out with respect to the conformal time $\eta = \int dt/a$. Equation (86) closely resembles the one dimensional Schrödinger equation in quantum mechanics

$$\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + [E - V(x)]\Psi = 0. \tag{87}$$

Comparing (87) and (86) we find that the role of the "potential barrier in space" $V(x)$ is played by the time dependent term $V(\eta) = -m^2a^2 + (1-6\xi)\ddot{a}/a$ which may be thought of as a "potential barrier in time" [82,178,84]. (The form of the barrier is shown in Fig. 14 assuming that Inflation is succeeded by radiative and matter dominated eras.) In quantum mechanics the presence of a barrier leads to particles being reflected and transmitted so that $\Psi_{in}(x) = \exp(ikx) + R(k)\exp(-ikx)$ in the incoming region, and $\Psi_{out}(x) = T(k)\exp(ikx)$ in the outgoing region. Similarly, the presence of the time-like barrier $V(\eta)$ will lead to particles moving forwards in time as well as backwards, after being reflected off the barrier. The scalar field at late times will therefore not be in its vacuum state ϕ_k^+ but will be described by a linear superposition of positive and negative frequency states

$$\phi_{out}(k,\eta) = \alpha\phi_k^+ + \beta\phi_k^- . \tag{88}$$

The role of reflection and transmission coefficients R, T is now played by the Bogoliubov coefficients α, β which quantify particle production and vacuum polarization effects and are obtained by matching 'in modes' during Inflation with 'out modes' defined during the radiation or matter dominated eras.

Due to the existence of space-time curvature, positive and negative frequencies can be defined only in the limiting case of small wavelengths, $\lim_{k\to\infty}\phi_k^\pm \simeq \frac{1}{\sqrt{2ka}}\exp(\mp ik\eta)$, for which effects of curvature can be neglected. The value of α, β is obtained by matching modes corresponding to the 'out' vacuum with those of the 'in' vacuum just after Inflation. (The 'in' and 'out' vacua are defined during Inflation and radiation/matter domination respectively.)

The net effect of particle creation and vacuum polarization is quantified by the vacuum expectation value of the energy-momentum tensor $\langle T_{ik}\rangle$. For $\xi < 0, |\xi| \ll 1$ and $m/H \lesssim 1$ the leading order contribution to $\langle T_{ik}\rangle$ is given by

$$\langle T_{ik}\rangle \simeq -\xi(R_{ik} - \frac{1}{2}g_{ik}R)\langle\Phi^2\rangle + \frac{1}{2}g_{ik}m^2\langle\Phi^2\rangle + ... \tag{89}$$

We immediately see that the first term is simply proportional to the Einstein tensor and the second has the covariant form usually associated with a cosmological constant (*i.e.* $T_{ik} = g_{ik}\Lambda$). Substituting for $\langle T_{ik}\rangle$ in the semiclassical Einstein equations

$$R_{ik} - \frac{1}{2}g_{ik}R = -8\pi G(T_{ik} + \langle T_{ik}\rangle)\ , \tag{90}$$

we find

$$3H^2 = 8\pi G(\rho_m + \rho_{vac}) \tag{91}$$

where

$$\rho_{vac} \equiv \langle T_{00}\rangle \simeq 3\xi H^2\langle\Phi^2\rangle + \frac{1}{2}m^2\langle\Phi^2\rangle \tag{92}$$

$$\langle\Phi^2\rangle = \frac{1}{2\pi^2}\int dk k^2|\phi_{out}(k,\eta)|^2. \tag{93}$$

The term proportional to $H^2\langle\Phi^2\rangle$ in (92) may be absorbed into the left hand side of (91) leading to

$$3H^2 \simeq 8\pi\bar{G}[\rho_m + \frac{1}{2}m^2\langle\Phi^2\rangle] \tag{94}$$

where $\bar{G} \simeq G/(1 + 8\pi G|\xi|\langle\Phi^2\rangle)$ is the new, time dependent gravitational constant. (Observational bounds on the rate of change of $\bar{G}$ set the constraint

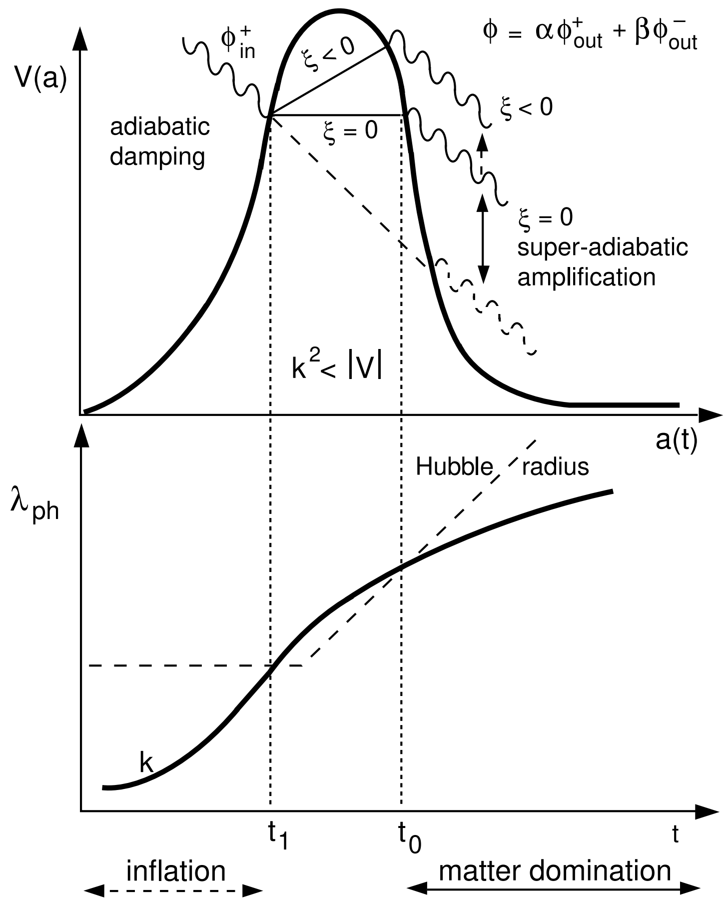

Fig. 14. The process of super-adiabatic amplification of zero-point fluctuations (particle production) is illustrated. The amplitude of modes having wavelengths smaller than the Hubble radius decreases conformally with the expansion of the universe, whereas that of larger-than Hubble radius modes freezes (if $\xi = 0$) or grows with time ($\xi < 0$). Consequently, modes with $\xi \leq 0$ have their amplitude super-adiabatically amplified on re-entering the Hubble radius after inflation (from Sahni & Habib 1998)(`the case` $\xi = 0$ `also describes quantum mechanical production of gravity waves in a FRW model [82].)`

$|\xi| \ll 1$.) As shown in [171] for $\xi < 0$ the value of $\langle\Phi^2\rangle$ can be very large, so that $\bar{G} \simeq 1/(8\pi|\xi|\langle\Phi^2\rangle)$ and

$$\begin{aligned} \Lambda_{eff} \equiv & \ 8\pi\bar{G}\langle T_{00}\rangle \simeq m^2/2|\xi|, \\ \Omega_\Lambda \equiv & \ \Lambda_{eff}/3H^2 \simeq \tfrac{1}{6|\xi|}(m/H)^2 \end{aligned} \tag{95}$$

We therefore find that the energy density of created particles defines an *effective cosmological constant* which can contribute significantly to the total density of the universe at late times leading to $\Omega_m + \Omega_\Lambda \simeq 1$ [171].

However, it should be noted that this result was obtained in the Hatree-Fock (or. semiclassical gravity) approximation (90) which is not exact in considerations of a single quantum field, since metric and field fluctuations may significantly deviate from their *rms* values. So, further study of this problem using stochastic methods (similar to those used in stochastic inflation [182,183,196] and stochastic reheating after inflation [115]) is desirable.

8. Phenomenological models of a dynamical Λ-term.

8.1. Overview.

Having, under the pressure of observational evidence and an aesthetical desire to keep the inflationary scenario of the early universe as simple as possible, admitted the existence of a constant positive Λ-term, it is natural to take a step beyond Einstein's original hypothesis and consider the additional possibility that the Λ-term is not an exact constant, but rather, describes a new dynamical degree of freedom (perhaps even a new form of matter). Really, neither observational data, nor inflationary considerations tell us that a "cosmological constant" is constant (though, as discussed above, it should change sufficiently slowly with time, in particular, slower than the Ricci tensor). In fact the effective Λ-term which appears in the inflationary scenario of the early universe is never an exact constant and rarely even an approximate constant of motion. (A recent analysis of observational data in the light of a time dependent Λ may be found in [203].)

To quantitatively describe this new degree of freedom (or a new form of matter), some phenomenological models of a dynamical Λ-term have to be introduced. The word "phenomenological" means that no attempt to derive these models from an underlying quantum field theory is being made, in contrast to examples discussed in previous sections. Historically, many phenomenological Λ-models were proposed since 1986 (not counting the "C-field" of Hoyle and Narlikar (1962) which was perhaps the earliest, though unsuccessful, attempt to introduce a dynamical Λ-term in cosmology). Depending upon their level of "fundamentality", these phenomenological methods may be classified into 3 main groups:

1) Kinematic models.
Here Λ is simply assumed to be a function of either the cosmic time t or the scale factor $a(t)$ of the FRW cosmological model.

2) Hydrodynamic models.
Here a Λ-term is described by a barotropic fluid with some equation of state $p_\Lambda(\rho_\Lambda)$ (dissipative terms may also be present).

3) Field-theoretic models.
The Λ-term is assumed to be a new physical classical field (which we shall call a *lambda-field*) with some phenomenological Lagrangian.

Of course, models from the last group are in a sense also the most fundamental. In particular, they may be used in a non-FRW setting. Additionally, their quantization is straightforward. However, if we restrict ourselves to a FRW model with small perturbations, the three different way of describing a Λ-term could lead to converging results. Note also, that it was recently proposed to call a dynamical Λ-term "quintessence" [24] (irrespective of the specifities of modelling, so this notion is wider than the notion of a *lambda-field*), though we don't consider it to be obligatory.

In the case of field-theoretic models, the most simple and natural is the model of a scalar field ϕ with some self-interaction potential $V(\phi)$, minimal coupling to gravity and no (or very weak) coupling to other known physical fields. The latter requirement follows not only from simplicity, but also from observational evidence (see [27] for a recent analysis of upper bounds on coupling of ϕ to the electromagnetic field). The assumption of minimal coupling to gravity may be relaxed, but only slightly (see [33] for constraints on the coefficient ξ in case of the $\xi R\phi^2/2$ coupling). Since the minimally coupled scalar field model has proven to be extremely successfuly in the case of the inflationary scenario, one might be tempted to use it for the description of a Λ-term. As a result, an overwelming part of recent theoretical activity has focussed on the scalar field model (more appropriately, on this class of models differing between themselves by the form of the scalar field potential $V(\phi)$).

8.2. When may a Λ-term be described by a minimally coupled scalar field?

Now let us consider the following important question: can a Λ-term *always* be described by a minimally coupled scalar field, for any observed behaviour of $a(t)$ or $H(a)$? To answer this question let us first consider the equations of motion describing a FRW universe with matter (dust) ρ_m and a scalar field ϕ

$$H^2 = \tfrac{8\pi G}{3}(\rho_m + \tfrac{\dot\phi^2}{2} + V), \;\; \rho_m = \tfrac{3\Omega_0 H_0^2}{8\pi G}(\tfrac{a_0}{a})^3, \tag{96}$$

$$\ddot\phi + 3H\dot\phi + \tfrac{dV}{d\phi} = 0,$$

$$\dot H = -4\pi G(\rho_m + \dot\phi^2). \tag{97}$$

The clue to whether a Λ-term can be successfully described by a minimally coupled field is provided by the background equation for $\dot H$ which we rewrite in the following form changing the independent variable from t to a ($\kappa = 0$ is assumed):

$$4\pi G a^2 H^2 \left(\frac{d\phi}{da}\right)^2 = -aH\frac{dH}{da} - \frac{3}{2}\Omega_m H_0^2 \left(\frac{a_0}{a}\right)^3 , \tag{98}$$

where a_0 is the present value of the FRW scale factor $a(t)$ and Ω_m includes all dust-like matter at present (CDM, baryons, sufficiently massive neutrinos, etc.). Since the left-hand side of Eq. (98) is always non-negative therefore so is the right-hand side. From this follows a fundamental restriction on the expansion law for the Universe, which we write in terms of the following inequality on the redshift dependence of the Hubble parameter $H(z)$, $1+z \equiv a_0/a$:

$$\frac{dH^2}{dz} \geq 3\Omega_m H_0^2 (1+z)^2 . \tag{99}$$

Actually, Eq. (99) is nothing more than the weak energy condition for a lambda-field: $\rho_\Lambda + p_\Lambda \geq 0$.

This inequality saturates in the case of a constant Λ-term (a cosmological constant). Equation (99) constitutes the necessary condition for an arbitrary $H(z)$ dependence to be physically described by a minimally coupled scalar field (in the absence of spatial curvature). It will be shown below that Eq. (99) is also a sufficient condition, since a knowledge of $H(z)$ and Ω_m permits a unique reconstruction of the self-interaction potential $V(\phi)$ of this scalar lambda-field (see section 8.4). Taken at $z = 0$, Eq. (99) reduces to the following relation between the acceleration parameter q_0 and Ω_m:

$$q_0 \leq \frac{3}{2}\,\Omega_m - 1 . \tag{100}$$

It should be emphasized that we have no idea at present whether or not Eqs. (99,100) are fulfilled. Only future observations will tell us that. Moreover, as was explained in previous sections, a constant Λ-term fits existing data very well. Thus, we know already that the inequalities (99,100) are close to saturation. So, it will be not an easy observational task. In this case, the presence of even a small spatial curvature may dramatically change our conclusions.

In the case of non-zero spatial curvature ($\kappa \neq 0$), Eq. (99) generalizes to:

$$\frac{dH^2}{dz} \geq 3\Omega_m H_0^2 (1+z)^2 - \frac{2\kappa}{a_0^2}\,(1+z) . \tag{101}$$

Therefore, if future data show that the inequality (99) is not valid, one has either to invoke a *positive* spatial curvature for the Universe ($\kappa = 1$), or else to discard this model entirely and to consider a more complicated model of a Λ-term, modelled by, say, a scalar-field non-minimally coupled to gravity. It is easy to verify that in the case of $\xi R\phi^2/2$ coupling, no necessary conditions such as (99) or (101) appear. However, as was mentioned above, this type of coupling is strongly restricted by observational data [33].

An interesting example of dissipationless decay of a lambda-field is provided by Peebles and Ratra (1988) who consider a minimally coupled scalar field rolling down a potential

$$V(\phi) = k/\phi^{\alpha},$$

subject to the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{102}$$

(k and α are constants, we set $M_P = 1$ for simplicity). Let us assume that the energy density of the scalar field

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{k}{\phi^\alpha} \tag{103}$$

is subdominant at early epochs (as demanded by CMB and nucleosynthesis constraints) so that $\rho_\phi < \rho_B$ at $z \gg 1$, where ρ_B is the density of background matter driving the expansion of the universe. Assuming a general expansion law for the universe $a(t) \propto t^q$ the field equation of motion (102) becomes

$$\ddot{\phi} + \frac{3q}{t}\dot{\phi} - \frac{\alpha k}{\phi^{1+\alpha}} = 0 \tag{104}$$

which has the solution

$$\phi \propto t^p, \quad p = \frac{2}{2+\alpha}. \tag{105}$$

Substituting ϕ in (103) we find $\rho_\phi \propto t^{2p-2}$, as a result if $p > 0$ the scalar field density ρ_ϕ decreases more slowly than the background density of matter or radiation which decreases as $\rho_B \propto t^{-2}$. Consequently we find

$$\frac{\rho_\phi}{\rho_B} \propto t^{\frac{4}{2+\alpha}} \tag{106}$$

i.e. for $\alpha > 0$ the scalar field density can dominate the matter/radiation density at late times even if it was subdominant to begin with [163,155]. (This attractive property of scalar fields is occasionally referred to as 'quintessence'.) The rate of growth of $\rho_\phi/\rho_{m,r}$ can be modulated by 'tuning' the value of α. Another way of arriving at this conclusion is to examine the equation of state of the scalar field while the latter is subdominant, this turns out to be

$$w_\phi = \frac{\alpha w_B - 2}{\alpha + 2} \tag{107}$$

where w_B is the background equation of state. From (107) we find $w_\phi < w_B$ *i.e.* the equation of state of the scalar field is less stiff than that of matter driving expansion. The conservation condition $\rho_{(\phi,B)} \propto a^{-3[1+w_{(\phi,B)}]}$ now guarantees that the scalar field will come to dominate the expansion dynamics of the universe even if it was initially subdominant. As a result ρ_ϕ can be significantly small during the radiation dominated epoch to satisfy nucleosynthesis constraints yet be large enough today to give rise to an accelerating universe in agreement with recent supernovae results. (Once ρ_ϕ begins to dominate the energy density, the universe enters into a period of accelerated expansion driven by the scalar field energy density which begins to mimic an effective Λ-term.)

A different possibility arises if we consider a scalar field rolling down an exponential potential

$$V(\phi) = V_0 \exp\left(-\lambda\phi/M_P\right).$$

In the case of a flat universe, the scalar field density scales exactly like the background density of matter driving the expansion of the universe so that the ratio of the scalar field density to the total matter density rapidly approaches a constant value [163,63,64]

$$\frac{\rho_\phi}{\rho_B + \rho_\phi} = \frac{3(1+w_B)}{\lambda^2} \tag{108}$$

($w_B = 0,\ 1/3$ respectively for dust, radiation). This 'tracker-like' quality whereby the scalar field contributes the same fixed amount to the total matter density allows it to play the role of a form of dark matter. However strong constraints on this model come from cosmological nucleosynthesis which suggests $\Omega_\phi \simeq \frac{\rho_\phi}{\rho_B} \lesssim 0.2$. [14] As a result the scalar field in these models is forever destined to remain subdominant, it can neither dominate the matter density of the universe nor give rise to its accelerated expansion rate.

A potential which interpolates between an exponential and a power law is

$$V(\phi) = V_0[\cosh \lambda\phi - 1]. \tag{109}$$

Since $V(\phi) \propto \exp \lambda\phi$, for $\lambda\phi \gg 1$, we would expect this potential to reproduce features of the exponential potential discussed earlier. As a result $\rho_\phi/\rho_B \simeq$ constant and $w_\phi \simeq w_B$, if the scalar field commences rolling from a large initial value. As the scalar field rolls down towards smaller values, the potential begins to resemble the Inflationary 'chaotic' form $V(\phi) \propto \lambda^2\phi^2$, leading to late-time Inflation during which $w_\phi \simeq -1$. Finally oscillations of the scalar field give rise to a 'dust-like' phase during which $w_\phi \simeq 0$.

[14] In a spatially closed universe the presence of an exponential potential can give rise to an intermediate 'coasting' epoch during which $a(t) = \alpha t$, where $\alpha \ll 1$. Such a universe bears great similarity to loitering models considered earlier, density perturbations grow faster during 'coasting' and the 'age problem' too can be resolved [169].

An unusual potential with interesting features was proposed in [215]

$$V(\phi) = V_0[e^{M_P/\phi} - 1]. \tag{110}$$

However, the requirement that $\Omega_\phi \ll 1$ during the matter dominated epoch, while $\Omega_\phi \sim 1$ nowadays, is fulfilled for this potential only if the present value of ϕ is significantly larger than M_{Pl}. Thus, for practical applications in the present universe, this potential shows little difference from the inverse-power-law potential $V \propto \phi^{-1}$.

A useful property of potentials (102), (109) & (110) is that they significantly alleviate the fine tuning problem associated with generating a small cosmological term at *precisely* the present epoch. As a result, ρ_ϕ can come to dominate the current cosmological density from a fairly general class of initial conditions. A phase space analysis of scalar field models was carried out in [163,63,64,125] where it was shown that both exponential and negative power-law potentials display appealing attractor-like qualities. However, despite the many attractive features of 'quintessence' models a degree of fine tuning does remain in fixing the parameters of the potential and has been commented on in [125,118].

It is worth pointing out in this context that the energy density of relic gravity waves created during Inflation (ρ_g) behaves like a tracker field since $\rho_g/\rho_B \simeq$ constant, if the expansion factor grows exponentially during Inflation [2]. For more realistic situations in which the inflaton field rolls down its potential slowly the ratio ρ_g/ρ_B increases with time with the result that the graviton energy density may become comparable to ρ_B at very late times provided Inflation commenced at the Planck epoch [168]. COBE measurements of the large angle anisotropy of the cosmic microwave background (CMB) however ensure that the gravity wave contribution to the total matter density is negligibly small today: $\Omega_g \lesssim 10^{-12}$ [178]. However the intriguing possibility that quanta of a different type of fundamental field (the dilaton perhaps) may come to dominate the energy density of the universe without necessarily violating CMB bounds remains to be investigated.

Some cosmological consequences of scalar field models and models with a decaying cosmological term have been analyzed in [155,75,140,177,36,69,70,23] [24,202,98,34,96,77,156,201,203]. Candidates for quintessence based on high energy physics and string theory are discussed in [35,118] and non-minimal scalar field models are treated in [74,33].

Phenomenological Λ models usually belong to the general category of models in which matter either violates or marginally satisfies the strong energy condition (SEC) $\rho + 3P \geq 0$. Scalar fields driving inflation as well as the models discussed earlier in this section furnish examples of matter which can violate the SEC. Other examples of such 'strange' or 'exotic' forms of matter include cosmic strings and domain walls. The field configuration within a string is in

Table 1
Summary of phenomenological Λ models. Here a is the scale factor, H the Hubble parameter, T the temperature, t the cosmic time (A, B, α are constants).

Evolutionary relation for $\Lambda(t)$	Reference
$\Lambda \propto t^{-2}$	[59,25,16,15,13,133,147]
$\Lambda \propto t^{-\alpha}$	[14,103,104]
$\Lambda \propto A + B\exp(-\alpha t)$	[14,181]
$\Lambda \propto a^{-2}$	[133,148,149,1,199,32,79,116]
$\Lambda \propto a^{-\alpha}$	[143,151,169,135,138,176,177] [93,102,147,193,23,24,202,96,77]
$\Lambda \propto \exp(-\alpha a)$	[162]
$\Lambda \propto T^{\alpha}$	[25,107]
$\Lambda \propto H^2$	[127,207,208,63,42]
$\Lambda \propto H^2 + Aa^{-\alpha}$	[3,28,172,200]
$\Lambda \propto f(H)$	[128,129]
$\dot{\Lambda} \propto g(\Lambda, H)$	[89,164]

the false vacuum state leading to $P = -\rho$ along the string length. A network of random non-intercommuting strings therefore possesses the average equation of state $P = -\rho/3$ which marginally satisfies the SEC [197]. The mean energy density of a string network dominated by straight strings decays as $\rho \propto a^{-2}$ leading to the linear expansion law $a \propto t$ [197,79]. Similarly $P = -\rho$ is satisfied along any two orthogonal directions within a domain wall leading to $P = -2\rho/3$ for a network of walls [197,180] and resulting in 'mild' Inflation $\rho \propto a^{-1}$, $a \propto t^2$. The presence of tangled strings and/or domain walls can be tested by measurements sensitive to the expansion dynamics of the universe. For instance recent supernovae results strongly suggest $w \lesssim -2/3$ which severely constraints the string network for which $w \simeq -1/3$. Thus it appears that a tangled network of strings is ruled out by current observations (see section 4.3).

A brief summary of some models with a decaying cosmological term is given in Table 1 (adapted from [147]), we should stress that most of these models are phenomenological and are therefore not necessarily backed by strong physical arguments.

Finally one should mention another phenomenological approach tied to the possibility of a cosmological term decaying and transferring its energy into particles and/or radiation [148,68]. Observationally such an approach can, in principle, be tested: in the case of dissipative, baryon number conserving decay of a Λ-term into baryons and antibaryons, the subsequent annihilation of matter and antimatter would result in a homogeneous gamma-ray flux which could be constrained by observations of the diffuse gamma-ray background in the Universe [68,138]. A decay of the cosmological term directly into radiation could be probed by cosmic microwave background anisotropies, cosmological nucleosynthesis etc. [68,174,17,146,138,151,176,177].

8.3. Relation between kinematic and dynamical descriptions of Λ

As pointed out in the previous section, although kinematic and dynamical models of Λ lie on completely different levels of fundamentality from the theoretical point of view, they may be equivalent if a background space-time is described by a FRW model. In particular, the simplest class of kinematic models

$$\Lambda \equiv 8\pi G\rho_\Lambda = f(a) \tag{111}$$

is then equivalent to hydrodynamic models based on an ideal fluid with the equation of state

$$p_\Lambda(\rho_\Lambda) = -\rho_\Lambda - \frac{1}{3}\frac{d\ln\rho_\Lambda}{d\ln a} \tag{112}$$

(with a being excluded from Eq. (112) using Eq. (111)).

Let us go further and present the correspondence between a popular subclass of these models where $\Lambda \propto a^{-\alpha}$ (or, equivalently, $p_\Lambda = \left(\frac{\alpha}{3} - 1\right)\rho_\Lambda$) and field-theoretical models for a minimally coupled lambda-field following [187] where the particular case $\alpha = 2$ (i.e. the Λ-term mimicking temporal behaviour of spatial curvature or non-relativistic cosmic strings) was considered. This gives an explicit example of the reconstruction of a lambda-field potential from $H(a)$. Now $\kappa = 0$ is assumed for simplicity, and we take $0 \le \alpha < 3$. The left inequality is necessary for the condition (99) to be satisfied, while the right inequality guarantees that $\rho_\Lambda \ll \rho_m$ during the matter-dominated stage while $z \gg 1$ (in addition, this condition makes p_Λ negative). In this case, the Hubble parameter $H(a)$ is given by

$$\frac{H^2}{H_0^2} = \Omega_m\left(\frac{a_0}{a}\right)^3 + (1-\Omega_m)\left(\frac{a_0}{a}\right)^\alpha\ . \tag{113}$$

Using the $0-0$ background Einstein equation and Eq. (98), the lambda-field potential $V(\phi)$ can be expressed in terms of $H(a)$:

$$8\pi G V(\phi) = aH\frac{dH}{da} + 3H^2 - \frac{3}{2}\,\Omega_m H_0^2\left(\frac{a_0}{a}\right)^3 \tag{114}$$

which reduces to

$$V = \frac{3 - \frac{\alpha}{2}}{8\pi G} H_0^2 (1 - \Omega_m) \left(\frac{a_0}{a}\right)^{\alpha} \tag{115}$$

for the case under consideration.

Now Eq. (98) may be integrated for the given $H(a)$ dependence to obtain

$$\frac{a}{a_0} = \frac{\Omega_m}{1 - \Omega_m} \sinh^{\frac{2}{3-\alpha}} \left((3 - \alpha) \sqrt{\frac{2\pi G}{\alpha}} (\phi - \phi_0 + \phi_1) \right) \tag{116}$$

where ϕ_0 is the present value of the lambda-field and

$$\exp\left((3 - \alpha) \sqrt{\frac{2\pi G}{\alpha}} \phi_1 \right) = \left(\frac{1 - \Omega_m}{\Omega_m}\right)^{\frac{3-\alpha}{2}} + \sqrt{1 + \left(\frac{1 - \Omega_m}{\Omega_m}\right)^{3-\alpha}} . \tag{117}$$

Finally, combining Eqs. (115, 116) we get an explicit expression for the interaction potential:

$$V(\phi) = \frac{(3 - \frac{\alpha}{2})(1 - \Omega_m)^{1+\alpha} H_0^2}{8\pi \Omega_m^{\alpha} G} \sinh^{-\frac{2\alpha}{3-\alpha}} \left((3 - \alpha) \sqrt{\frac{2\pi G}{\alpha}} (\phi - \phi_0 + \phi_1) \right) . \tag{118}$$

At early times during the matter-dominated stage, this potential is an inverse power-law ($V(\phi) \propto (\phi - \phi_0 + \phi_1)^{-\frac{2\alpha}{3-\alpha}}$) (we do not consider here what happens with $V(\phi)$ even earlier, during the radiation-dominated stage). While during the current, Λ-dominated epoch, it changes its form to an exponential. This shows why the assumptions of a purely power-law dependence of Λ on a or, equivalently, of a linear equation of state $p_\Lambda = w_\Lambda \rho_\Lambda$, $w_\Lambda = const$ are not "natural": they require fine-tuning between the present value of the lambda-field ϕ_0 and the value of ϕ where the potential changes its form. On the other hand, neither can this possibility be ruled out completely.

In addition, this example of reconstruction of $V(\phi)$ shows that, in field-theoretic models of Λ based on a minimally coupled scalar field, there is no lower limit on the present value of w_Λ other than -1 (which follows from the weak energy condition (99)). The opposite statement in [215,188] is a consequence of a number of additional assumptions (equipartition of energy densities of all fields including the lambda-field at the end of inflation, use of a subclass of possible initial contitions whose solutions for ϕ have reached an intermediate asymptote which they call the "tracker" solution by the present time, consideration of some special classes of potentials), none of which is obligatory. In particular, a "tracker" solution may have w_Λ arbitrarily close to -1 at present, if an inverse power-law potential with a small exponent is used.

In view of the large number of models capable of predicting a small cosmological constant at the present epoch, it is necessary to ask whether cosmological observations themselves may be used to determine model parameters uniquely. The answer to this question is (fortunately) in the affirmative, at least for the class of minimally coupled scalar field models discussed earlier. This is easily demonstrated by considering Equations (114) and (98) which express $V(\phi)$ and ϕ in terms of the Hubble parameter H and its first derivative dH/dz. Consequently one can determine the form of the potential $V(z)$ (mimicking the Λ-term) if the Hubble parameter $H(z)$ is known from observations. There are two independent methods for determining $H(z)$. The first is related to the luminosity distance d_L, discussed in section 4.2 [186,101]. From (28) we easily find

$$H(z) = [\frac{d}{dz}\{\frac{d_L(z)}{1+z}\}]^{-1}. \tag{119}$$

Thus the luminosity distance $d_L(z)$ determines the Hubble parameter $H(z)$ uniquely ! Now (98,114) can be used to reconstruct the form of the potential $V(z)$ (or $V(\phi)$) and the equation of state $w_\phi(z)$ in a model independent manner. (However, what is required for an unambiguous determination of $V(\phi)$ is the present matter density Ω_m[186].) Formula (119) can also be used for an unambiguous determination of $H(z)$ from the angular-size distance $d_A(z)$ introduced in section 4.5, if we use the relation $d_A(z) = d_L(1+z)^{-2}$ [187].

Another means of determining $H(z)$ is associated with the growth of linear density fluctuations responsible for the formation of large scale structure [185,186]. The growth of linearized perturbations in a collisionless medium has the well known form

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_m\delta = 0 \tag{120}$$

where the value of H is determined from (97). (On scales $\ll 200\mathrm{h}^{-1}$ Mpc the Λ-field is practically unclustered and can be treated as a smooth component if $|m_\phi^2| \equiv |d^2V/d\phi^2| \lesssim H_0^2$). Although it is not possible to solve (120) analytically for an arbitrary potential $V(\phi)$, the inverse problem of determining H once δ is known is exactly solvable ! We demonstrate this by first performing a change of variables $t \to a,\ d/dt \to aHd/da$ which reduces (120) to a first order linear differential equation for H^2:

$$a^2\frac{d\delta}{da}\frac{dH^2}{da} + 2\left(a^2\frac{d^2\delta}{da^2} + 3a\frac{d\delta}{da}\right)H^2 = 3\Omega_0 H_0^2(\frac{a_0}{a})^3\delta. \tag{121}$$

Equation (121) has the exact solution

$$H^2 = \frac{3\Omega_0 H_0^2 a_0^3}{a^6}\left(\frac{d\delta}{da}\right)^{-2} \int\limits_0^a a\delta \frac{d\delta}{da} da = 3\Omega_0 H_0^2 \frac{(1+z)^2}{\delta'^2} \int\limits_z^\infty \frac{\delta|\delta'|}{1+z} dz, \qquad (122)$$

where $\delta' = d\delta/dz$. Setting $z = 0$ in this expression, we arrive at a very interesting relationship between Ω_0 and $\delta(z)$:

$$\Omega_0 = \delta'^2(0)\left(3\int\limits_0^\infty \frac{\delta|\delta'|}{1+z} dz\right)^{-1}. \qquad (123)$$

Substituting this relationship in (122) we finally obtain

$$H(z) = H(0)\left[\frac{(1+z)^2\delta'^2(0)}{\delta'^2(z)} - 3\Omega_0 \frac{(1+z)^2}{\delta'^2(z)} \int\limits_0^z \frac{\delta|\delta'|}{1+z} dz\right]^{\frac{1}{2}}. \qquad (124)$$

Clearly knowing δ and δ' we can determine $H(z)$ and hence $V(z)$. For very large z, $z \gg 1$, observational difficulties make it unlikely that δ will be known to great accuracy, at least in the near future. However in this regime the flat matter dominated solution $\delta \propto (1+z)^{-1}$ provides a very good approximation since $\Omega_m \to 1$ for $z \gg 1$.

It should be pointed out that the above method of reconstructing the Λ-term potential from observations is complementary to that used to reconstruct the inflaton potential [126]. Whereas the luminosity distance d_L or the growth rate of the linearized density contrast $\delta(z)$ can be used to reconstruct $V(\phi)$, the inflaton potential is reconstructed on the basis of the primordial amplitude and spectrum of relic density perturbations and gravity waves created during inflation (also see [141]).

9. Universality of Λ and anthropic arguments for its small value.

In this, the final section of our review, we must ask the following question: do we expect the present value of Λ to be fundamental (= defined by the parameters of a physical theory) or accidental (= determined by initial conditions in the early Universe)? At present, we have no answer to this question. Models for Λ considered in previous sections admit both possibilities. For instance, in the class of minimally-coupled lambda-field models with an inverse power-law potential, the present value of Λ is fundamental (= defined by the parameters of $V(\phi)$ only) if "initially" (at the end of inflation or at a later moment when the lambda-field becomes a separate degree of freedom of matter) ϕ was sufficiently small, so that the corresponding solution for $\phi(t)$ had time to reach a future attractor (the "tracker" solution of [215,188]) by the present epoch.

On the other hand, if the initial value ϕ_{in} is large, then the present value $\phi_0 \approx \phi_{in}$, and the current value of the Λ-term is accidental. Note that in the latter case Λ is practically time independent now. Such a large value of ϕ_{in} may, for instance, be generated during an early inflationary stage, in which case stochastic methods [182,183,196] may be used to derive probability distributions for ϕ_{in} and Λ. As a byproduct of such a mechanism, small quasi-static inhomogeneous perturbations of Λ will also be generated. [15]

If Λ is accidental, then a wide range of "explanations" for its currently (small) value can be given, based on the most reliable form of the anthropic principle - the weak anthropic principle. However, even if Λ is fundamental and can be expressed through other microphysical constants, one may still try to use a more controversial form of this principle - the strong anthropic principle. [16]

An anthropic argument for $\Lambda > 0$ has been suggested by Banks (1985) and Weinberg (1987), who felt that the extraordinary difference between likely values of the vacuum energy $\rho_\Lambda \sim \rho_m \sim 10^{-29}\mathrm{g/cm}^3$ and the expected value (from a consideration of Planck scale physics) $\rho_P \sim 10^{93}\mathrm{g/cm}^3$ could only be understood through anthropic arguments, since, in the absence of a fundamental symmetry which set the value of Λ to precisely zero, it would be extremely fortuitous if particle physics determined a value for ρ_Λ which was comparable to the matter density at *this precise* moment in the history of the universe. The case for the anthropic principle as a viable means for understanding properties of the universe has received a strong measure of support from recent developments in inflationary cosmology. A self-consistent treatment of quantum effects in inflationary models has shown that the entire universe may consist of an ensemble of sub-universes (separated from each other by particle horizons) having 'all possible types of vacuum states and all possible types of compactification' of extra space-time dimensions [131]. According to this picture our observable universe is but one of an infinite number of universes each having its own set of conserved quantities and dimensions. Since in each

[15] Previous discussions involving quantum cosmology also held the possibility that the value of Λ is not determined uniquely. For instance Hawking (1984) showed that the wave function for the universe could contain a superposition of terms with different values for the cosmological constant. Investigating the effect of wormholes on quantum gravity, Coleman (1988a,b) subsequently showed that coupling constants whose values were not fixed by symmetries in the Lagrangian could take on all possible values in the superposition of terms describing the state vector in quantum cosmology.

[16] The weak anthropic principle in the narrow sense states that our location in space and time should be such that it admits the existence of intelligent life. An extension of this principle is that initial conditions allow the existence of such a region in space-time. On the other hand the strong anthropic principle states that laws of nature should permit the existence of intelligent life. It may be noted that the border between these two versions of the anthropic principle is not absolutely rigid. Namely, by generalizing a physical theory (say, the electroweak model) with fixed constants into a more general theory where these constants may have arbitrary values depending upon initial conditions, we make a step from the strong to the weak anthropic principle (also see [11]).

sub-universe physical fields determining the value of Λ have distinct values it is reasonable to expect that the value of Λ varies from one sub-universe to another.

Weinberg (1987) showed that large values of Λ were unlikely to be 'observed' since the presence of observers demanded the existence of galaxies and galaxy formation was strongly suppressed if the energy in the cosmological constant greatly exceeded the matter density (also see [55,198]). Martel, Shapiro & Weinberg (1998) have suggested that the probability that observers living in a given sub-universe will measure a value ρ_Λ for the 'vacuum energy' be given by the expression

$$\mathcal{P}(\rho_\Lambda) = \frac{F(\rho_\Lambda)}{\int_0^\infty F(\rho_\Lambda) d\rho_\Lambda} \tag{125}$$

where $F(\rho_\Lambda)$ is the fraction of matter in galaxies in a sub-universe with vacuum energy $\rho_\Lambda = \Lambda/8\pi G$ [137]. The value of $F(\rho_\Lambda)$ is calculated assuming Gaussian initial fluctuations at recombination, with a COBE-normalized cold dark matter spectrum with a cosmological constant (ΛCDM). The requirement that the observed value $\Omega_{\Lambda,*}$ in our sub-universe equal the statistical mean or median evaluated over all sub-universes (*i.e.* $\Omega_{\Lambda,*} = \langle\Omega_\Lambda\rangle$, where $\Omega_\Lambda = \Lambda/3H_0^2$) gives a value which peaks in the region $\Omega_{\Lambda,*} \sim 0.6 - 0.9$ for a broad region of parameter space and assuming fairly reasonable conditions for galaxy formation [137]. Thus small observed values of Ω_Λ appear to be strongly disfavoured by the anthropic argument !

10. Summary and Discussion

In the absence of a symmetry in Nature which would set the value of the cosmological constant to precisely zero, one is forced to either set $\Lambda = 0$ by hand, or else look for mechanisms that can generate $\Lambda = \Lambda_{\rm obs} > 0$, where $\Lambda_{\rm obs} \sim 10^{-29}$g cm^{-3} is the value of the Λ-term inferred from recent supernovae observations. We have discussed several mechanisms which could, in principle, give rise either to a time independent cosmological *constant*, or else a time dependent Λ-term. To the former category primarily belong models which associate Λ with a property of the vacuum such as the vacuum energy associated with symmetry breaking, or vacuum polarization and particle production effects in curved space-time. Mechanisms predicting a time dependent Λ take their cue from Inflation and generate a time varying Λ out of scalar fields rolling down a potential. Models with a fixed Λ run into fine-tuning problems since the ratio of the energy density in Λ to that of matter/radiation must be tuned to better than one part in 10^{60} during the early universe in order that $\Lambda/8\pi G \simeq \rho_{\rm matter}$ today. Scalar field models considerably alleviate this problem though some fine-tuning does remain in determining the 'correct choice' of parameters in the scalar field potential.

It has been known for several years that the flat FRW ΛCDM cosmological model with an approximately flat spectrum of initial adiabatic perturbations fits observational data better and has a larger admissible region of parameters (H_0, Ω_m) than any other cosmological model with both inflationary and non-inflationary initial conditions (see, e.g., [113,114,144,184,5]). For instance according to a typical expert opinion made several years ago "for $H_0 > 60$ $\mathrm{km\,s^{-1}\,Mpc^{-1}}$, this model is probably the *only* feasable model" [184]). Now, with new data on high redshift type Ia supernovae becoming available, we are closer than ever to concluding that this is *the right* cosmological model (at least to a first approximation) even if $H_0 < 60$. Moreover, using type 1a supernovae data and with improved data on gravitational clustering at high redshifts soon expected, we may progress further and investigate whether Λ depends weakly on time.

Turning to the observational situation, constraints on the cosmic equation of state arise from observations at: low redshifts (age of universe, cluster abundances, baryon fraction, velocity fields, etc.), intermediate redshifts (ages of distant galaxies & QSO's, angular size vs. redshift, gravitational lensing, Type 1a supernovae, the Lyman α forest etc.) and high redshifts (cosmic microwave background). Each set of observations has its own systematic errors and although considerable progress has been made in trying to understand systematics it is safe to say that at any given time at least one set of observations is likely to be well off the mark !

Of the low redshift tests, the age of the universe, cluster abundances and baryon fraction all appear to favour a low density universe, with $\Omega_m \lesssim 0.3$ in clustered matter. A tone of dissonance is however provided by recent observations of the angular size of compact radio sources which seem to suggest a critical density matter dominated universe, although evolutionary effects clearly need to be better understood before a strong case for $\Omega_m \simeq 1$ is made based on these results alone.

The strongest support for an accelerating universe comes from intermediate redshift results for Type 1a supernovae. At the time of writing close to a hundred supernovae have been analyzed by two teams: The Supernova Cosmology Project and the High-Z Supernova Search Team, both teams getting mutually consistent results for $\{\Omega_m, \Omega_\Lambda\}$. It should be pointed out that the supernovae results do not by themselves pick out a flat universe from other possibilities; a cursory look at fig. (6) shows that a closed universe with $\Omega_m + \Omega_\Lambda > 1$ appears preferred although a flat universe is also accommodated by current observations. However the combined likelihood analysis of Sn1a + CMB observations strongly supports a flat universe with $\Omega_m + \Omega_\Lambda \simeq 1$, primarily due to the presence of a Doppler peak in the CMB data at intermediate angular scales $\theta \sim 1^\circ$. Thus although observations do seem to suggest that the universe may be spatially flat with a large fraction of its density in the form of a cosmological Λ-term, it may be premature to rule out, on the basis of current data alone, models that are spatially open or even matter dominated and flat.

Great progress is however expected on the observational front in the coming 5 - 10 years. Conservative estimates suggest that one should expect over ~ 50 new Type 1a events to be added to the supernovae inventory every year (including several at significantly higher redshifts than $z \sim 1$). Thus by the time of the launch of the MAP and PLANCK satellites (during 2001 & 2007 respectively) one would expect our understanding of supernovae related parameter estimation to have improved by over an order of magnitude. Since both MAP and PLANCK missions are expected to pinpoint the location and amplitude of the first Doppler peak at the level of a few percent accuracy, they should provide a decisive answer to the question of whether or not we live in a critical density universe. The definitive answer to the question of whether the universe is flat and accelerating may therefore have to wait just a few more years !

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