OBSERVATIONAL SELECTION BIAS AFFECTING THE DETERMINATION OF THE EXTRAGALACTIC DISTANCE SCALE

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ABSTRACT

The influence of Malmquist bias on the studies of extragalactic distances is reviewed, with brief glimpses of the history from Kapteyn to Scott. Special attention is paid to two kinds of biases, for which the names Malmquist biases of the first and second kind are proposed. The essence of these biases and the situations where they occur are discussed.

The bias of the first kind is related to the classical Malmquist bias (involving the "volume effect"), while the bias of the second kind appears when standard candles are observed at different (true) distances, whereby magnitude limit cuts away a part of the luminosity function. In particular, study of the latter bias in distance indicators such as Tully Fisher, available for large fundamental samples of galaxies, allows construction of an unbiased absolute distance scale in the local galaxy universe where approximate kinematic relative distances can be derived. Such investigations, using the method of normalized distances or of the Spaenhauer diagram, support the linearity of the Hubble law and make it possible to derive an unbiased value of the Hubble constant.

1. INTRODUCTORY REMARKS

A red line in the history of astronomy is the extension of distance measurements into deep space as a result of efforts to take advantage of what Nature offers astronomers. Ptolemy (ca 150) stated that "none of the stars has a noticeable

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parallax (which is the only phenomenon from which distances can be derived),” which illustrates the situation of when we see a class of celestial bodies and also have in mind a sound method but cannot apply it because of insufficient observational means. Of course, only after Copernicus was there strong theoretical pressure to search for stellar parallaxes as a crucial cosmological test and as a method of distance determination. However, long before this triangulation method was successfully applied by Bessel, Henderson, and Struve in the 1830s (see Hoskin 1982), the photometric method, based on the inverse square law of light flux from a point source introduced by Kepler, had been recognized as a possible way of getting information on stellar distances. In 1668, James Gregory applied it to the distance of Sirius, using the Sun as the calibrator star. In Gregory’s method of “standard candles,” one had to assume that stars are other suns, identical to our Sun, and are observed through the transparent Euclidean space where Kepler’s inverse square law is valid.

1.1 *Stars and Galaxies are Gathered from the Sky and not from Space*

The above notes illustrate how determination of distances is intimately related to our general astronomical knowledge and assumptions, and also to our abilities to measure the directions and fluxes of weak photon streams. Though knowledge and observational methods, and hence construction of the distance ladder, are steadily advancing, there are fundamental difficulties that will always haunt the measurers of the universe.

The astronomer makes observations from a restricted vantage point in time and space. In fact, he or she does not observe celestial bodies in space, but in the sky (as traces on photographic plates or CCD images). Fortunately, very luminous objects exist that may be detected even from large distances. This diversity in the cosmic zoo, which allows one to reach large distances and hence makes cosmology possible, also involves problematic aspects.

First, from large distances only highly luminous objects are detectable, and the photons usually do not carry information on how much the objects differ from their average properties: There are no genuine Gregory’s standard candles. Second, objects in the sky that are apparently similar, i.e. have the same distance modulus, actually have a complicated distribution of true distances. Distances larger than those suggested by the distance modulus are favored because of the volume ($r^2 dr$) effect. Third, at large distances there is much more space than within small distances from our position. Because very luminous objects are rare, these are not found in our vicinity. We can see objects that as a class might be useful indicators of large distances but perhaps cannot make the crucial step of calibration, which requires a distance ladder to reach the nearest of such objects (Sandage 1972). As an extreme example, even if one could find

standard candles among luminous quasars (Teerikorpi 1981), there is no known method to derive their distances independently of redshift.

When one uses "standard candles" or "standard rods," calibrated on a distance ladder, systematic errors creep into the distance estimates. When related to the above problems, they are often collectively called Malmquist-like biases.

According to Lundmark's (1946) definition, a distance indicator is a certain group of astronomical objects that has the same physical properties in different galaxies. The principle of uniformity of natural law is assumed to be valid when one jumps from one galaxy to another. Slightly expanding Lundmark's definition, one may say that a distance indicator is a method where a galaxy is placed in three-dimensional space so that its observed properties agree with what we know about galaxies, their constituents, and the propagation of light. An ideal distance indicator would restrict the galaxy's position somewhere in a narrow range around the true distance if that indicator has an intrinsic dispersion in its properties, and a set of such indicators should lead to a consistent picture of where the galaxies are. In practice, even a "good" indicator is affected by sources of systematic error due to the intrinsic dispersion.

1.2   *Distances and Hubble Law*

The Hubble law between distance and cosmological redshift is a blessing for the cosmographer. A great motivation for investigation of the distance scale, it is also helpful for tackling the problems mentioned above.

Systematic errors in obtained distances are often recognized as a deviation from the linear Hubble law, and the reality and speed of galaxy streams, for example, closely depend on how well distances to galaxies are known. When one speaks about the choice of one or another distance scale, this is intimately connected with the Hubble constant H_o. In the Friedmann model, H_o together with q_o allows one to extend the distance scale to high cosmological redshifts where classical distance indicators are lost. However, this extension is not the topic of my review [for discussion of such questions, see e.g. Rowan-Robinson (1985)].

To discuss biases in extragalactic distances, one might like to know what "distance" represents. As McVittie (1974) says, distance is a degree of remoteness; in some sense or another, faint galaxies must be remote. Only a cosmological model gives the exact recipe for calculating from the observed properties of an object its distance (which may be of different kinds). Because our basic data consist of directions and fluxes of radiation, we are usually concerned with luminosity or diameter distances. An error, say, in the luminosity distance in a transparent space means that if one puts a genuine standard candle beside the distance indicator in question, its photometric distance modulus is not the same.

Among the variety of distance concepts, one would like to think that there exists a fundamental one, corresponding to meter sticks put one after another

from the Sun to the center of a galaxy. For instance, in the Friedmann universe, the theoretical and not directly measurable "momentary" proper distance is often in the background of our minds as the basic one (Harrison 1993), and the luminosity and other types of distances are the necessary workhorses. This review refers to the local galaxy universe where the different distance types are practically the same; in any case, the tiny differences between them are overwhelmed by selection biases and other sources of error. Another allowance from Nature is that in this distance range, evolutionary effects can be forgotten: In an evolving universe, distance indicators often require that the look-back time should be much less than the global evolutionary time scale.

1.3 *Aim of the Present Review*

In the recent literature, comprehensive and technical discussions of various aspects of the Malmquist bias exist (Willick 1994, Strauss & Willick 1995, Hendry & Simmons 1995). The reviews by Sandage (1988a, 1995) put the subject in the general context of observational cosmology. I have in mind a wide audience, from the general astronomer to the quantum cosmologist, who may need an introduction to the Malmquist biases. This problem is so characteristic of the special nature of astronomy that it should be included in introductory astronomy text books. In the otherwise versatile work on the cosmological distance ladder by Rowan-Robinson (1985), the problem of Malmquist bias receives only a passing note, and the extensive review article on distance indicators by Jacoby et al (1992) is also relatively silent in this respect.

The concepts of the different kinds of Malmquist biases are rather simple, though it is possible to dress the discussions in mathematics "complicated enough to be but dimly understood except, perhaps, by their authors" as Sandage (1988a) noted. On the other hand, one sometimes sees references to "the well-known Malmquist bias," which is usually a sign of an insufficient treatment of this problem. The present review is written in the spirit of a useful middle path.

I do not discuss in detail different distance indicators but concentrate on the central issues of the biases. Also, this is not a discussion of the Hubble constant, though it occasionally appears. I also leave aside the question of local calibration. Of course, the biases discussed here are not the only problems of the distance scale. For instance, the supposed distance indicators may not at all fulfill Lundmark's definition, an example of which is Hubble's brightest stars in distant galaxies that were actually HII regions (Sandage 1958).

I take examples mostly from the Tully-Fisher (TF) indicator, which is the most widely applied method in the local universe, with samples of several thousands of galaxies and the calibration of which can now be based on an increasing number of Cepheid distances. Understanding the biases in such large and fundamental samples will always be the test bench on which to build

the distance scale and where differing alternative scales must be ultimately compared.

The theory of how to deal with the selection effects affecting distances has gradually evolved to such a level that one may already speak about an emerging general theory of distance indicators. Relevant results are often scattered in articles concerned with a variety of topics, and though I have tried to find a representative reference list, I apologize if some interesting aspects go unnoticed. Finally, I should admit that my several years of pleasant collaboration with the Meudon and Lyon groups must somewhat "bias" this review.

2.   SOME HISTORY FROM KAPTEYN TO SCOTT

Of course, this is not a proper place to write a history of selection biases, or how they have been invented and reinvented, considered, or neglected in astronomical works of the present century. However, it seems helpful to introduce the reader to the current discussion of this subject by picking from the past a few important fragments.

2.1   *Kapteyn's Problem I and Problem II*

In a paper on the parallaxes of helium stars "together with considerations on the parallax of stars in general," Kapteyn (1914) discussed the problem of how to derive the distance to a stellar cluster, presuming that the absolute magnitudes of the stars are normally distributed around a mean value. He came upon this question after noting that for faint stars the progress of getting kinematical parallaxes is slow and "can extend our knowledge to but a small fraction of the whole universe." A lot of magnitude data exist for faint stars, but how to put them to use? He formulated Problem I as follows:

> Of a group of early B stars, all at practically the same distance from the sun, we have given the average apparent magnitude $\langle m \rangle$ of all the members brighter than m_o. What is the parallax π of the group?

Changing a little notation and terminology, Kapteyn's answer to this question may be written as an integral equation, where the unknown distance modulus μ appears:

$$\langle m \rangle = \frac{\int_{-\infty}^{m_1} m \exp\left[-(m - \mu - M_o)^2 / 2\sigma^2\right] dm}{\int_{-\infty}^{m_1} \exp\left[-(m - \mu - M_o)^2 / 2\sigma^2\right] dm}. \tag{1}$$

Because values for the parameters M_o and σ of the gaussian luminosity function are known, one may solve the distance modulus μ. Note that the integration over apparent magnitudes is made from $-\infty$ to m_1, the limiting magnitude. Kapteyn calculated a table for practical use of his equation, so that from the

observed value of $\langle m \rangle$, one gets the distance modulus μ. If one simply uses the mean absolute magnitude M_o and calculates the distance modulus as $\langle m \rangle - M_o$, a too-short distance is obtained. Though Kapteyn did not discuss explicitly this bias, his method was clearly concerned with the Malmquist bias of the second kind, a typical problem in photometric distance determinations (see Section 3).

Kapteyn recognized that the situation is different if stars are scattered at varying distances, leading to his Problem II:

> Of a group of early B stars, ranging over a wide interval of distance, given the average apparent magnitude of all the stars brighter than m_o we require the average parallax of the group.

In this scenario, if one now uses Kapteyn's table mentioned above, taking the μ corresponding to $\langle m \rangle$, one generally obtains an incorrect average distance modulus $\langle \mu \rangle$. Also, as in Problem I, one cannot take $\langle \mu \rangle = \langle m \rangle - M_o$, either. This Kapteyn's problem (for which he did not offer a complete solution) is related to what is called the classical Malmquist bias.

In Problem I, one has information on relative distances; in this particular case the distances are equal. The necessity of having relative distances indicates that the solution to Problem I is not applicable to a "group" of one star. In Problem II there is no a priori information on relative distances. On the other hand, in order to calculate a mean distance modulus, one needs such information as must be extracted from the only data available, i.e. from the distribution of apparent magnitudes. Again, a sample of a single star with $m = m_o$ cannot be a basis for solving Problem II (as an answer to the question "what is the most probable distance of this star?"), unless one makes some assumption on how the magnitudes of the other stars are distributed.

2.2   *The Classical Malmquist Bias*

In his work, "A study of the stars of spectral type A," Malmquist (1920) investigated how to derive the luminosity function of stars from their proper motions, provided that it is gaussian and one knows the distribution of apparent magnitudes up to some limiting magnitude. This led Malmquist to investigate the question of what is the average value (and other moments) of the quantity R, or the reduced distance, as earlier introduced by Charlier:

$$R = 10^{-0.2M}. \tag{2}$$

Malmquist made three assumptions: 1. There is no absorption in space. 2. The frequency function of the absolute magnitudes is gaussian (M_o, σ). 3. This function is the same at all distances. The third assumption is the principle of uniformity as implied in Lundmark's (1946) definition of distance indicators.

Using the fundamental equation of stellar statistics, Malmquist derived $\langle R^n \rangle$ and showed that it may be expressed in terms of the luminosity function constants M_o and σ and the distribution $a(m)$ of apparent magnitudes, connected with the stellar space density law. Especially interesting for Malmquist was the case $n = -1$, or the mean value of the "reduced parallax," that appears in the analysis of proper motions. However, for distance determination, the case $n = 1$ is relevant because it allows one to calculate, from the mean value of the reduced distance, the average value of the distance $\langle r \rangle$ for the stars that have their apparent magnitude in the range $m \pm 1/2 \, dm$ or their distance modulus $\mu = m - M_o$ in the range $\mu \pm 1/2 \, d\mu$. The result is, written here directly in terms of the distance modulus distribution $N(\mu)$ instead of $a(m)$,

$$\langle r \rangle_\mu = r(\mu) \exp(0.5 b^2 \sigma^2) N(\mu + b\sigma^2)/N(\mu), \tag{3}$$

where $b = 0.2 \cdot \ln 10$. This equation is encountered in connection with the general Malmquist correction in Section 6. Naturally, in Malmquist's paper one also finds his formula for the mean value of M for a given apparent magnitude m:

$$\langle M \rangle_m = M_o - \sigma^2 d[\ln a(m)]/dm. \tag{4}$$

The term including the distribution of apparent magnitudes (or distance moduli in Equation 3) reduces to a simple form when one assumes that the space density distribution $d(r) \propto r^{-\alpha}$:

$$\langle M \rangle_m = M_o - (3 - \alpha) \cdot 0.461 \, \sigma^2. \tag{5}$$

With $\alpha = 0$, one finally obtains the celebrated Malmquist's formula valid for a uniform space distribution:

$$\langle M \rangle_m = M_o - 1.382 \, \sigma^2. \tag{6}$$

Hubble (1936) used Malmquist's formula (Equation 6) when, from the brightest stars of field galaxies (and from the magnitudes of those galaxies), he derived the value of the Hubble constant. Hubble derived from a local calibrator sample the average (volume-limited) absolute photographic magnitude and its dispersion for the brightest stars. As "the large-scale distribution of nebulae and, consequently, of brightest stars is approximately uniform," he derived the expected value for the mean absolute magnitude of the brightest stars $\langle M_s \rangle$ for a fixed apparent magnitude. His field galaxies were selected, Hubble maintained, on the basis of the apparent magnitudes of the brightest stars, which justified the calculation and use of $\langle M_s \rangle$. In the end, he compared the mean apparent magnitudes of the brightest stars in the sample galaxies with $\langle M_s \rangle$, calculated the average distance $\langle r \rangle$, and derived the value of the Hubble constant (526 km/s/Mpc). Hence, it is important to recognize that this old value

of H_o, canonical for some time, already includes an attempt to correct for the Malmquist bias. Also, it illustrates the role of assumptions in this type of correction (what is the space density law, what is the mode of selection of the sample, etc).

2.3 *The Scott Effect*

Forty years ago, Scott (1957) published an important paper on "The Brightest Galaxy in a Cluster as a Distance Indicator." Looking back to this study, one may find it contained many basic points that were later discussed in connection with the Malmquist bias of the second kind. Her concern was how the availability of a distant cluster of galaxies influences the use of its brightest galaxy as a standard candle. Availability means that (*a*) at least *n* cluster members must be brighter than the limiting magnitude m_1 of the plate, and (*b*) the apparent magnitude of the brightest galaxy must be brighter than another limit m_2, which is needed for measurements of magnitude and redshift. Let us pick up a few conclusions of Scott (1957, p. 249):

- (A)t any given distance, a cluster with many members is more likely to be available to the observer than a cluster containing fewer galaxies, or

- if a very distant cluster is available to the observer, then this cluster must be unusual, and

- the brightest galaxies actually observed in very distant clusters must have a tendency to possess brighter absolute magnitudes than the average brightest galaxies in the nearer clusters, hence,

- for distant clusters, the simple use of the brightest galaxy in the cluster as a distance indicator leads to an underestimate of the distance.

Scott used both numerical simulations and an analytical model to show the size of systematic errors that the condition of availability causes to the derived distance at different true distances, and she concluded that the selection effect is bound to influence seriously the Hubble m-z diagram constructed for brightest cluster galaxies. She (1957, p. 264) also concluded that

- an interpretation of the deviations from linearity in the magnitude redshift relation that occur near the threshold of available instruments cannot be made with confidence without appropriate allowance for selection bias.

She (1957, p. 264) also pondered about how to make a difference between a real and selection-induced deviation from linearity in the Hubble law:

- (T)he question as to whether or not the apparent deviation from linearity is an effect of selection [or a real effect] may perhaps be solved by the accumulation of further data using instruments corresponding to larger values of both m_1 and m_2.

Of the above two extracts, Hubble's application is clearly concerned with the classical Malmquist bias (or Kapteyn's Problem II), whereas the discussion by Scott has a relation to Kapteyn's Problem I, though the effect in galaxy clusters is more complicated. In fact, there is no mention of the classical Malmquist bias in Scott's paper and really no need for it.

3.   TWO KINDS OF MALMQUIST BIAS

In recent years, Kapteyn's Problems I and II have been discussed (by authors unaware of that early work) in connection with extragalactic distances from methods like Tully-Fisher (TF) and Faber-Jackson, for determination of the Hubble constant and peculiar velocity maps. Both of these problems have been sometimes referred to as Malmquist bias. I have collected in Table 1 examples of the used nomenclature.

3.1   *Biases of the First and Second Kind*

In this review, I call these two problems Malmquist bias of the first and second kinds: The first kind is the general Malmquist bias, directly connected with the classical treatment by Malmquist (1920). One might briefly define these two aspects as follows:

- Malmquist bias of the first kind is the systematic average error in the distance modulus μ for a class of galaxies with "derived" $\mu = \mu_{\text{der}} = $ constant.

- Malmquist bias of the second kind is the systematic error in the average derived $\langle \mu_{\text{der}} \rangle$ for the class of galaxies with "true" $\mu = \mu_{\text{true}} = $ constant.

In discussions of the first bias, one is interested in the distribution of true distance moduli for the constant derived modulus, whereas the second bias is concerned with the distribution of derived distance moduli at a constant true

Table 1   Various names used for the two kinds of biases

Author	Nomenclature	
Kapteyn (1914)	Problem II	Problem I
Landy & Szalay (1992)	General Malmquist bias	—
Han (1992)	Geometry bias	Selection effect
Teerikorpi (1990, 1993)	"V against r"	"r against V"
Sandage (1994a)	Classical	Distance-dependent
Hendry & Simmons (1994)	Bayesian	Frequentist
Willick (1994)	Inferred-distance problem	Calibration problem
Strauss & Willick (1995)	Malmquist bias	Selection bias
Teerikorpi (1995)	M-bias of the first kind	M-bias of the second kind

distance (e.g. at a constant redshift, if the Hubble law is valid). These two biases have appeared under different names. Teerikorpi (1990, 1993) separated the study of the Hubble diagram into "distance against velocity" and "velocity against distance." Hendry & Simmons (1994) spoke about "Bayesian" and "frequentist" approaches in terms of mathematical statistics, whereas Willick (1994) emphasizes "inferred-distance problem" and "calibration problem." In his bias properties series, Sandage (e.g. 1994a) also made a clear difference between the classical Malmquist bias and the distance-dependent effect, and he showed in an illuminating manner the connection between the two.

More recently, Strauss & Willick (1995) have used the terms Malmquist bias and selection bias, whereby they emphasize by "selection" the availability of galaxies restricted by some limit (flux, magnitude, angular diameter). On the other hand, one may say that the first kind of Malmquist bias is also caused by a selection effect dictated by our fixed position in the universe and the distribution of galaxies around us, and in the special case of the inverse relation (Section 5), it also depends on the selection function.

The first kind of bias is closely related to the classical Malmquist (1920) bias; hence the name is well suited. However, Lyden-Bell (1992) prefers to speak about Eddington-Malmquist bias: Malmquist (1920) acknowledged that Eddington had given Equation 6, which corresponds to the special case of constant space density, before he had. As for the second bias, the situation is not so clear. However, in Malmquist (1922), in his Section II.4, one finds formally the equations needed for the calculation of the second bias, when the luminosity function has a gaussian distribution in M. Malmquist does not seem to have this application in mind, but in view of the widespread habit of speaking about the Malmquist bias (or Malmquist effect) in this connection, it might be appropriate to name the distance-dependent effect after him as well [another possibility would be to speak about Behr's (1951) effect, cf Section 3.3].

3.2 Situations Where the First or General Malmquist Bias Appears

Let us consider a class of galaxies having a gaussian luminosity function $G(M_p, \sigma)$ acting in the role of a standard candle. Such a class may be defined, e.g. via a fixed value of the TF parameter $p = \log V_{max}$, and it is assumed that the calibration has been made, i.e. the volume-limited value of M_p is known. A sample of these galaxies with magnitudes measured up to some limit gives us a collection of derived distance moduli $\mu = m - M_p$. If in addition the redshifts are available, one may construct the μ-log V Hubble diagram, and try to determine the Hubble constant H from the following expected relation:

$$\log V = \log H - 0.2\mu + \text{const.} \tag{7}$$

If the error in the distance modulus has a gaussian distribution $G(0, \sigma)$, one can take simple averages $\langle \log V \rangle$ and $\langle \mu \rangle$, and solve for $\langle \log H \rangle$. In reality, there is a systematic error in each subaverage $\langle \mu_i \rangle$ ($\mu_i = $ constant), and this systematic error generally depends on the value of μ_i if the space distribution of galaxies is not uniform. The error $\Delta \mu(\mu_i)$ is given by Malmquist's formula (Equation 3). One point should be especially emphasized: The ratio $N(\mu + b\sigma^2)/N(\mu)$ refers to the distance moduli in the ideal case when the sample is not influenced by a magnitude limit, i.e. when $N(\mu)$ contains information on the real space distribution of the galaxies.

One implication of the general Malmquist bias is that the Hubble diagram does not necessarily show a linear Hubble law, because the bias depends on the real distribution of galaxies and is different for different (derived) distances. Also, in different directions of the sky, one may derive by the mentioned method different values of the Hubble constant, possibly interpreted, but incorrectly, to be caused by our own peculiar motion or by large scale streams of galaxies. Such explanations have been applied to the Rubin-Ford effect (Sandage & Tammann 1975a,b, Fall & Jones 1976, James et al 1991), the supergalactic anisotropy (Teerikorpi & Jaakkola 1977), and the Great Attractor velocity field (Landy & Szalay 1992).

The general Malmquist correction has in recent years been discussed particularly in connection with the attempts to derive maps of peculiar velocities or galaxy streams. The radial component of a peculiar velocity for a galaxy with true distance r and observed correct radial velocity V_{obs} is as follows:

$$V_{pec} = V_{obs} - H_o \cdot r. \tag{8}$$

Let us look at galaxies in some direction, with their derived distances within a narrow range $r_{der} \pm 1/2 dr$. Perhaps these galaxies are also in the real space so close by that they have a common peculiar velocity (bulk flow or stream), though different cosmological components. Then $\langle V_{obs} \rangle - H_o \cdot \langle r_{der} \rangle$ does not give the correct peculiar velocity, unless a proper correction for $\langle r_{der} \rangle$ is made, i.e. the Malmquist correction of the first kind, which gives the correct average distance for the galaxies in the $r_{der} \pm 1/2 dr$ range.

Lynden-Bell et al (1988) used such a correction in their Great Attractor paper, though they assumed a uniform distribution of galaxies. This means that for their diameter (vs velocity dispersion) indicator for elliptical galaxies, the bias formula is analogous to the classical Malmquist formula:

$$\langle r \rangle = r_{est} \left(1 + 3.5\sigma_D^2 \right). \tag{9}$$

3.3   *Situations Where the Second Malmquist Bias Is Important*

A simple way to see the second kind of Malmquist bias in action is to take the sample above, calculate log H for each galaxy, and plot log H against redshift.

As demonstrated in several studies during the last twenty or so years, log H stays first roughly constant and then starts to increase. So clear and dramatic is this phenomenon, and also so much expected from simple reasoning as due to bias, that it is appropriate to repeat the words of Tammann et al (1980): "If an author finds H_o to increase with distance he proves in the first place only one thing, i.e., he has neglected the Malmquist effect! This suspicion remains until he has proved the contrary." To this statement, one should add that here "distance" means the true distance or at least a true relative distance (e.g. redshift in the case of the Hubble law). If "distance" is the measured or inferred distance, H_o does not necessarily change with the distance, though it may have a wrong average value—we have come back to the first kind of bias. This also serves as a warning that simple comparison of the "linearity" of photometric distances from two methods may hide a common bias, as seen in the comparisons of two or more distances by separate indicators discussed by de Vaucouleurs (1979). Each of the indicators suffers the same type of bias properties.

It seems that Behr (1951) was the first to point out, after comparison of the width of the Local Group luminosity function to that of the field galaxies, that application of the standard candle method may lead to systematically short distances at large true distances. As we discussed above, this idea was transformed into a quantitative model by Scott (1957) for the selection of brightest cluster galaxies. It was then reinvented by Sandage & Tammann (1975a) and Teerikorpi (1975a,b), in connection with concrete field samples of spiral galaxies with van den Bergh's (1960a,b) luminosity classes. The basic reasoning is illustrated by the formula that connects the derived distance modulus to apparent magnitude, the assumed standard absolute magnitude M_p, and to the magnitude limit m_{lim}:

$$\mu_{der} = m - M_p \leq m_{lim} - M_p. \tag{10}$$

Clearly, there is a maximum derived distance $\mu_{max} = m_{lim} - M_p$. However, because the standard candle actually has a dispersion σ in absolute magnitude, some galaxies can be seen from true distances beyond μ_{max} that necessarily become underestimated.

Under the assumption of a gaussian luminosity function $G(M_p, \sigma)$ and a sharp magnitude limit m_{lim}, it is a straightforward task to calculate the magnitude of the second Malmquist bias $\langle \mu \rangle - \mu_{true}$ at each true distance modulus (Teerikorpi 1975b; Sandage 1994a), or how much $\langle \log H \rangle$ will increase due to the magnitude limit cutting galaxies away from the fainter wing of the luminosity function. Such an analytical calculation needs the correct distance scale and the relevant dispersion σ as input, which restricts its application in practice. However, it gives the general behavior of the bias, which is quite similar to the observed behavior of $\langle \log H \rangle$ vs kinematic distance, and forms the basis for recognition and correction methods that are independent of H_o (Section 4).
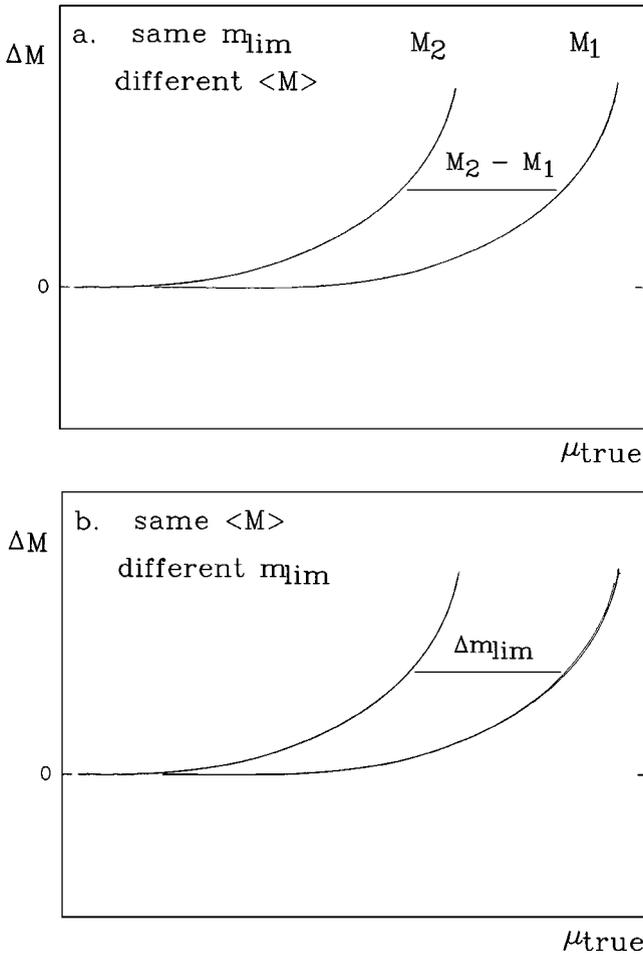
Figure 1 Illustrations of how the bias ΔM in the average magnitudes of standard candles depends on true distance modulus μ_{true}: a. When the limiting magnitude m_{lim} is fixed, the bias curves are simply shifted by $M_2 - M_1$. b. Increasing m_{lim} shifts the bias curve of a single standard candle class by Δm_{lim} in μ. At small true distances, one sees the unbiased plateau.

A fundamental property of the theoretical bias curves is that the curves for standard candles with different means, M_p, will show a shift along the axis of true distance (redshift) (see Figure 1*a*). This, in the first place, led to the recognition of the bias by Teerikorpi (1975a), where different van den Bergh's luminosity classes were inspected, and by analogy, to the proposal that

de Vaucouleurs's parabolic velocity–distance relation (de Vaucouleurs 1972) was caused by a related effect.

An earlier example of the second Malmquist bias at work, is found in Hawkins (1962), where the composite Hubble diagram for field galaxies (using the magnitudes as distance indicators) was interpreted as supporting the quadratic law $z = k \cdot r^2$. This was suggested to be expected from the gravitational redshift in a static uniform universe, contrary to the linear law expected for the expansion redshift in a homogeneous and isotropic universe, described by the Robertson-Walker line element (Robertson 1955). If written formally as a velocity law, the quadratic law assumes the form $\log H = 0.5 \log V + const$, which roughly describes the run of the data points between the unbiased region ($\log H = 0 \cdot \log V + const$) and the strongly biased one ($\log H = 1.0 \cdot \log V + const$). One sometimes sees, though not quite appropriately, "Lundmark's law" mentioned in connection with the quadratic law. True, Lundmark (1924, 1925) produced the first diagram where a dependence between redshift and distance was discernable, and he suggested the representation $z = a + b \cdot r - c \cdot r^2$. In Lundmark's formal solution, the negative quadratic term does not seem to be related to any selection or other real effect—the scatter from galaxy diameters as distance indicator was large, and he might have easily suggested a linear law. Perhaps he was motivated by the interesting prediction that redshifts have a maximum value (3000 km/s), which was soon to be contradicted by Humason's measurements.

A special case of the second kind of Malmquist bias is related to Kapteyn's Problem I and sometimes called cluster population incompleteness bias. Here the galaxies are at the same true distance, and the bias in the derived distance modulus is caused and calculated similarly as above. This bias makes the distances to clusters of galaxies, calculated by the TF method, progressively too short, and as it does for field galaxies, making log H increase with true distance. Assertions in literature (e.g. Aaronson et al 1980, de Vaucouleurs & Corwin 1985, Bothun et al 1992) have stated that there is no Malmquist bias in clusters (because the galaxies in clusters have no volume effect like the field galaxies), implying that one may utilize the TF method without bias. However, there were indications in the 1980s that the clusters, in comparison with Malmquist corrected field galaxies, give H_o values that are too large, which led to recognition of the cluster incompleteness bias by Teerikorpi (1987), Bottinelli et al (1987), and Kraan-Korteweg et al (1988).

4.   ATTEMPTS TO OVERCOME THE SECOND KIND OF BIAS

The solution, Equation 1, proposed by Kapteyn for his Problem I, gives the distance to a cluster or to any subsample of galaxies known, e.g. from the

Hubble law or other velocity field model, to be at the same distance. However, this presupposes that (*a*) there is a sharp magnitude limit, with data complete up to m_{lim}, (*b*) the standard candle has a gaussian distribution $G(M_p, \sigma)$, (*c*) the mean M_p is known from local considerations (calibration), and (*d*) the dispersion σ is known. These are rather strict conditions. Fortunately, there are situations and aims that do not necessarily require a complete knowledge of all these factors. For instance, study of the linearity of the Hubble law does not need an absolute calibration of the standard candle, and knowledge of σ is not necessary in all methods for deriving the Hubble constant.

4.1 *The Bias-Free Redshift Range*

Sandage & Tammann (1975a) introduced the concept of the bias-free distance (redshift) range in their Hubble parameter H-vs-log V_o diagram for luminosity classified spiral galaxies, which showed an increase of H with redshift. They explained this increase as caused by the truncation effect of the limiting magnitude, which makes the derived distances too small.

One expects an unbiased region at small true distances (redshifts) because the sample can be distance-limited rather than flux-limited, hence no part of the faint end of the luminosity function is truncated at appropriately small distances. It is only here that the Hubble constant can be derived without correction for bias. Such a region of about constant H was clearly visible in the H-vs-log V_o diagram of Sandage & Tammann (1975). As a first approximation, they cut away galaxies more distant than a fixed V_o ($\approx$2000 km/s), independently of the morphological luminosity class. Actually, each luminosity class has its own limiting distance, as Sandage & Tammann recognized. Using one fixed V_o, one (*a*) loses high-V_o data, part of it possibly unbiased, and (*b*) allows a remaining bias due to intrinsically faint luminosity classes with their proper limit $\langle V \rangle_o$. This was inspected by Teerikorpi (1976), where from the ST data (and their luminosity class calibration) the low value of $H_o = 41$ was derived, in comparison with $H_o = 57$ by Sandage & Tammann (1975). I give this reference because it was a step toward the method of normalized distances, later applied to samples of galaxies with TF measurements. The similarity with $H_o \approx 43$ by Sandage (1993) is probably not just a coincidence; both determinations relied on M101. An up-to-date discussion of the bias in the luminosity class method was given by Sandage (1996a).

4.2 *The Method of Normalized Distances for Field Galaxies*

Assume that we have two standard candle (galaxy) classes, each having gaussian luminosity functions $G(M_1, \sigma)$ and $G(M_2, \sigma)$, hence simply shifted in M by $\Delta M = M_1 - M_2$. If both are sampled up to a sharp magnitude limit m_{lim}, it is easy to see that in the bias vs true distance modulus μ diagram, the bias of the second kind suffered by these two candles is depicted by curves of the same

form but separated horizontally by constant $\Delta\mu = -\Delta M$ (see Figure 1a). The curve of the brighter candle achieves only at larger distances the bias suffered by the fainter candle already at smaller distances. In this way, simultaneous inspection of two or more standard-candle classes gives a new dimension to the problem of how to recognize a bias. Figure 1b shows another important property of the bias behavior. If one keeps the standard candle the same but increases the limiting magnitude by Δm_{lim}, the bias curve shifts to larger distances by $\Delta\mu = \Delta m_{lim}$. This is the basis of what Sandage (1988b) calls the "adding of a fainter sample" test.

van den Bergh's morphological luminosity classes clearly showed this effect (Teerikorpi 1975a,b), and even gave evidence of the "plateau" discussed below. In the beginning of 1980s, extensive studies started to appear where the relation (Gouguenheim 1969, Bottinelli et al 1971, Tully & Fisher 1977) between the magnitude (both B and infrared) and maximum rotational velocity of spiral galaxies V_{max} was used as a distance indicator. In the following, the direct regression (M against fixed $\log V_{max} \equiv p$) form of this TF relation is written as

$$M = a \cdot p + b. \tag{11}$$

There was at that time some uncertainty about which slope to use in distance determinations to individual galaxies—direct, inverse, or something between? Then Bottinelli et al (1986) argued that in order to control the Malmquist bias of the second kind, it is best to use the direct slope, so that the regression line is derived as M against the fixed observed value of p, without attempting to correct the slope for the observational error in p. The observed value of p devides the sample into separate standard candles analogous to Malmquist's star classes. A similar conclusion was achieved by Lynden-Bell et al (1988) in connection with the first kind of bias.

The importance of the direct relation is the fact that this particular slope allows one to generalize the mentioned example of two standard candles to a continuum of p values and in this way to recognize and investigate how the Malmquist bias of the second kind influences the distance determinations [that such a bias must exist also in the TF method was suspected by Sandage & Tammann (1984) and introduced on a theoretical basis in Teerikorpi (1984)]. If one inspects the whole sample (all p values clumped together) in the diagnostic $\log H$ vs d_{kin} diagram (cf Figure 1), the bias may not be very conspicuous. On the other hand, if one divides the sample into narrow ranges of p, each will contain a small number of galaxies, which makes it difficult to see the behavior of the bias for each separate standard candle within $p \pm 1/2dp$. For these reasons, it is helpful to introduce so-called normalized distance d_n (Teerikorpi 1984, Bottinelli et al 1986), which transforms the distance axis in the $\log H$ vs d_{kin} diagram so that the separate p classes are shifted one over the other and the
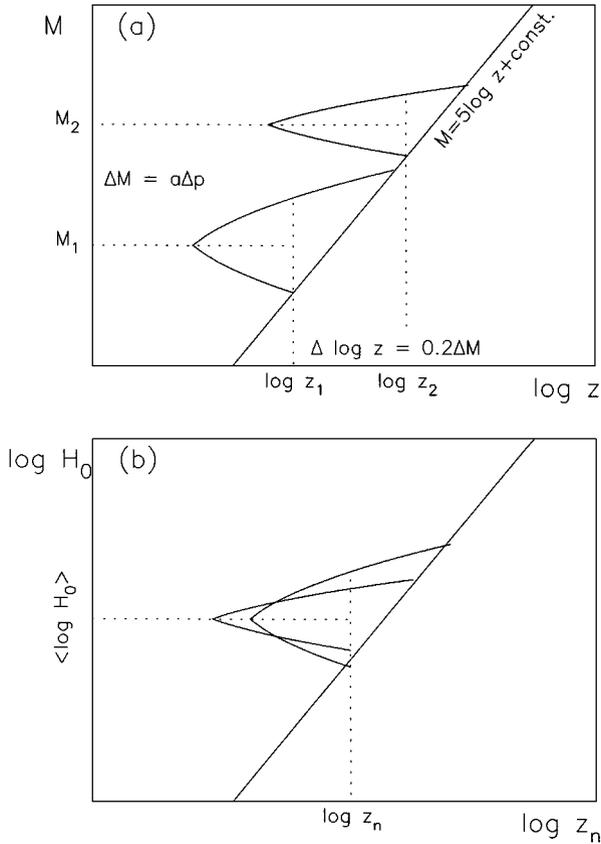
Figure 2   Schematic explanation of how the method of Spaenhauer diagrams (the triple-entry correction or TEC) and the method of normalized distances (MND) are connected. Upper panel shows two Spaenhaeur (M vs log z) diagrams corresponding to TF parameters p_1 and p_2. The inclined line is the magnitude limit. When the data are normalized to the log H vs $\log z_n$ diagram, using the TF slope a, the separate Spaenhauer diagrams "glide" one over the other and form an unbiased plateau that, among other things, can be used for determination of the Hubble constant.

bias behavior is seen in its purity (see Figure 2 in this regard):

$$\log d_n = \log d_{kin} + 0.2(a \cdot p + b - \text{const}). \tag{12}$$

In fact, one might also term this transformed distance as the effective one. This method of normalized distances (MND) uses as its starting point and test bench an approximative kinematical (relative) distance scale (d_{kin}, e.g. as provided by the Hubble law or Virgo-centric models) used with observed redshifts. The

method usually investigates the bias as seen in the Hubble parameter log H, calculated from the (direct) TF distance for each galaxy using the (corrected) radial velocity. If the Hubble law is valid, i.e. there exists a Hubble constant H_o, then one expects an unbiased plateau at small normalized distances, a horizontal part from which the value of H_o may be estimated from the plot of the apparent H_o vs d_n. Bottinelli et al (1986) applied the method to a sample of 395 galaxies having B magnitudes and the TF parameter p and could identify clearly the plateau and determine H_o from it. Certain subtle points of the method were discussed by Bottinelli et al (1988a), who also presented answers to the criticism from de Vaucouleurs & Peters (1986) and Giraud (1986). A somewhat developed version of it has recently been applied to the KLUN (kinematics of the local universe) sample constructed on the basis of the Lyon-Meudon extragalactic database and containing 5171 galaxies with isophotal diameters D_{25} (Theureau et al 1997).

4.3   *Spaenhauer Diagrams and the Triple-Entry*
Correction by Sandage

The triple-entry correction (TEC) of Sandage (1994a,b) is an approach and method for bias recognition, derivation of unbiased TF relations, and calculation of unbiased Hubble constant, which in some respects differs from the method of normalized distances though is based on similar basic reasoning. The theory of the TEC method is given in a clear manner in the two articles by Sandage (1994a,b), and it is applied to the Mathewson et al (1992a,b) sample of 1355 galaxies in Federspiel et al (1994). The method, originally suggested by Sandage (1988b,c), is based on Spaenhauer (1978) log V_o-M diagrams, which reveal how the average absolute magnitude M of a standard candle changes with increasing kinematical distance V_o when the magnitude limit cuts away a progressively larger part of the luminosity function.

Because these two approaches are the most developed ways of deriving the direct TF relation and the value of H_o from large field spiral samples, it is important to see clearly how they are connected. This is best done with the aid of the absolute magnitude M-vs-log V_o diagram (Figure 2, *upper panel*), similar to Figure 3 by Sandage (1994b), though with only two inserted Spaenhauer patterns corresponding to TF parameters p_1 and p_2. The inclined straight line corresponds to the magnitude limit, cutting away everything to the right of the line. The tips, i.e. the apex ⟨M⟩, of the Spaenhauer patterns give the average TF magnitudes M_1 and M_2 for p_1 and p_2, respectively.

Of course, the data used in the MND can be given exactly the same Spaenhauer representation as in Figure 2. Then what happens when the data are tranformed into the form used in the MND? First, calculate the Hubble parameter log H for each galaxy, using an approximate TF relation. This shifts the tips of the Spaenhauer patterns to about the same vertical level in the log H-vs-log V_o
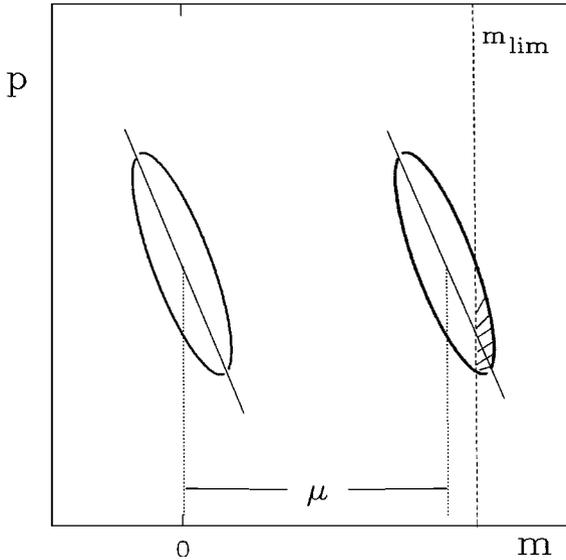
Figure 3 Schematic explanation of how the inverse TF relation (p vs M) may under ideal conditions overcome the Malmquist bias of the second kind. The nearby calibrator sample is made to glide over the distant sample so that the regression lines overlap. The condition for this is given by Equation 14, with $\mu = \mu_{est}$.

diagram. Second, shift the kinematical distances (log V_o) of the p_2 class by the factor $0.2(M_2 - M_1) = 0.2a(p_2 - p_1)$. This normalization puts the bias curve for p_2 on top of the bias curve for p_1, and the Spaenhauer patterns lie one over the other (Figure 2, *lower panel*). The individual unbiased parts of the patterns thus amalgamate together to form the common unbiased plateau. Refering to Figure 2, the MND means letting the upper Spaenhauer pattern glide down along the limiting magnitude line until it settles over the lower pattern.

    In both methods, the TF relation $M = a \cdot p + b$ is derived from the unbiased data: in the TEC using the undistorted part of the Spaenhauer pattern and in the MND using the unbiased plateau in an iterative manner. A difference is that in the MND, the dispersion σ is generally assumed to be a constant for all p, whereas the TEC allows different σs for different p classes. The same basic idea lies behind the two, with TEC leaving the p classes separated where their individual behavior is better inspected, whereas MND unites them into one ensemble, so that the collective and common bias behavior is more easily seen. With the large KLUN sample now available, it is also possible with MND to conveniently cut the sample into many subsamples according to type, inclination, etc, in order to investigate in detail their influence. In this manner, the inclination correction and the type dependence have been studied

by Bottinelli et al (1995) and Theureau et al (1996), respectively. An indication of the close relationship between the MND and TEC are the similar values of H_o that were derived from field samples by Theureau et al (1997) and Sandage (1994b), of 55 ± 5 and 48 ± 5 km/s/Mpc.

Finally, it should be noted that MND does not depend on any assumption on the space density distribution, as sometimes has been suspected. An advantage of both MND and TEC is that they are relatively empirical in essence: A minimum of assumptions is needed. More analytical methods, though needed in some situations, rely on ideal mathematical functions and assumed behavior of the selection function (Staveley-Smith & Davies 1989; Willick 1994) that often are not met.

Sandage (1994a) subtitled his paper "The Hubble Constant Does Not Increase Outward." He emphasized that inspection of the Spaenhauer diagrams of M_{cal} vs $\log \upsilon$ for samples with different limiting magnitudes and different TF parameter p values allows one to exclude any such significant systematic deviation from the linear Hubble law as repeatedly proposed by Segal (cf Segal & Nicoll 1996) with the quadratic redshift-distance law of his chronogeometric cosmology. Especially, addition of information in the form of several standard candles breaks the vicious circle, which allows one to interpret the data either in terms of the linear Hubble law plus selection bias or the quadratic law plus little selection bias (meaning very small dispersion of the luminosity function). The same thing can be said of the method of normalized distances: One would not expect the constant plateau for H, built by galaxies of widely different p values and widely different redshifts, if H_o actually does not exist. The method is also the same as adding a fainter sample of the same type of indicator and testing if the bias properties of the extended sample moves toward fainter magnitudes by the difference in the magnitude limits of the two lists (Sandage 1988b) (see Figure 1*b*).

4.4   *The Cluster Incompleteness Bias*

When one constructs the log H-vs-p diagram for members of a galaxy cluster using the direct TF relation, one generally observes log H increasing towards smaller values of the TF parameter p, i.e. towards the approaching magnitude limit. In one distant cluster, with a small number of measured galaxies, this trend may be difficult to recognize, and it is especially hard to see the expected unbiased plateau at large values of p, which gives the correct distance (and log H). Using data from several clusters and normalizing, instead of distance, the parameter p to take into account the vicinity of the magnitude limit, the combined log H-vs-p_{norm} diagram reveals better the bias behavior. This was shown by Bottinelli et al (1987) using 11 clusters (B-magnitude TF relation), and furthermore by Bottinelli et al (1988b) for clusters with infrared data. In

both cases, the distance scale and the Hubble constant were derived in good agreement with the method of normalized distances for field galaxies. Kraan-Korteweg et al (1988) and Fouqué et al (1990) analyzed the distance of the Virgo cluster and clearly also gathered evidence of the influence of the cluster incompleteness bias.

A graphic way of determining the distance to a cluster (in fact, solving Kapteyn's formula of Problem I for an ensemble of clusters and deriving the value of H_o) was used in Bottinelli et al (1988b). Willick (1994) has described an iterative approach.

Recently, Sandage et al (1995) gave a very illuminating exposition of the cluster incompleteness bias, using field galaxies in redshift bins to imitate clusters at different distances. Again, using the principle that adding a fainter sample shows the presence of bias, if it exists, from their Spaenhauer approach, these authors show the presence and behavior of the bias in the imitated clusters. An important empirical result concerns how deep in magnitudes, beyond the brightest galaxy, one must penetrate a cluster so as to avoid the bias. This depends on the dispersion σ of the TF relation and on the value of the TF parameter p. For small values of p (slowest rotators, faintest galaxies), the required magnitude range may be as wide as 8 mag. It is clear that for distant clusters such deep samples are beyond reach, and one must always be cautious of the incompleteness bias because it always leads to too large a value of H_o if uncorrected.

Another important point emphasized by Sandage et al (1995) is the artificial decrease in the apparent scatter of the TF relation, if derived from magnitude-limited cluster samples. This was also pointed out by Willick (1994), and it was implicit in the conclusion by Bottinelli et al (1988a) that this bias changes the slope of the TF relation. Sandage et al (1995) calculate the dependence of the apparent σ on the magnitude range Δm reached for a cluster. For instance, if the true dispersion is 0.6 mag, penetration by $\Delta m = 3$ mag gives $\sigma \approx 0.4$ mag, and $\Delta m = 6$ mag is needed to reveal the true σ. They conclude that the sometimes-mentioned small dispersions for the TF relations below $\sigma \approx 0.4$ mag are probably caused in this way (for other viewpoints, see Willick et al 1996, Bernstein et al 1994).

The problem of artificially decreased scatter is a dangerous one because it may lead to the conclusion that the selection bias, which is generally dependent on σ^2, is insignificant. In this manner, the bias itself produces an argument against its presence.

4.5 *Normalization: Other Applications*

Normalization is not restricted to kinematic distances of field galaxies or to TF parameters of cluster members. It is a useful approach to try when a range of values exists for some parameter on which the bias depends. The aim is to

reveal the bias from the observed trend and to recognize the unbiased plateau. Teerikorpi (1986) inspected how selection affects the inner rings in galaxies as a distance indicator by constructing a normalized parameter from the quantity k, which indicates the dependence of the inner ring size on de Vaucouleurs's morphological galaxy class, as calibrated by Buta & de Vaucouleurs (1983). The analysis dropped Buta & de Vaucouleurs's $H_o = 93 \pm 4$ to 75 ± 3 [or to 58 for the Sandage & Tammann (1975a) primary calibration], which is close to what happens with the TF indicator. There are two factors at work: 1. A relation between ring size and galaxy luminosity makes the former larger when the latter is larger. 2. At large distances, small rings cannot be measured. These are akin to a problem inherent in HII region size as a distance indicator, which is also clearly seen by normalization (Teerikorpi 1985). The possibilities of the inner ring method of Buta & de Vaucouleurs (1983) do not yet seem to be fully exploited.

5.   THE INVERSE TULLY-FISHER RELATION AND THE SECOND KIND OF BIAS

de Vaucouleurs (1983) differentiated between what he called the Malmquist effect (the progressive truncation of the luminosity function at increasing distances in a magnitude-limited sample) and the Malmquist bias in the distances derived from such a sample. One may intuitively think that if there is a way of classifying galaxies into absolute magnitude bins, for example, by using de Vaucouleurs's luminosity index or the TF relation $M = ap + b$, the Malmquist effect, as defined above, will certainly cut away fainter galaxies from the sample, but then the parameter p "glides" simultaneously. de Vaucouleurs argues that this compensates for the systematic distance dependent effect. However, the theory of the Malmquist bias of the second kind in direct TF distance modulus shows that such a compensation is not complete: Average p glides to larger values, but still, no matter what the value of p is, the corresponding distribution of true M is cut at a common M_{lim} that depends only on the distance. One cannot escape this fact, which means that in the observed sample, the distance indicator relation $\langle M \rangle = ap + b$ is necessarily distorted and causes the second kind of Malmquist bias. However, at each distance the bias is smaller by the factor $\sigma^2/(\sigma^2 + \sigma_M^2)$, as compared with the simple truncation effect of the luminosity function with dispersion σ_M.

5.1   *The Ideal Case of the Inverse Relation*

In the ideal case, the TF parameter p is not restricted by any such observational limit as M_{lim}. Hence, at any distance, the distribution of observed p corresponding to a fixed M, and especially its average $\langle p \rangle_M$, is the same. Schechter (1980)

thus realized that the inverse relation

$$p = a'M + b' \tag{13}$$

has the useful property that it may be derived in an unbiased manner from magnitude-limited samples, if there is no selection according to p. He used this relation in a study of the local extragalactic velocity field, which requires that kinematic distances minimize the p residuals (see also Aaronson et al 1982).

In what manner could one use the inverse relation as a concrete distance indicator? Assume that there is a cluster of galaxies at true distance modulus μ. Derive the distance modulus for each galaxy i that has p_i measured, using the inverse relation as a "predictor" of M: $\mu_i = m - (1/a')(p - b')$. Teerikorpi (1984) showed that the distance estimate $\langle \mu_i \rangle$ is unbiased, under the condition that there is no observational restriction to p. This result was supported by numerical simulations in Tully (1988).

Our ordinary way of thinking about distance indicators is closely linked to the direct relation: Measure p, determine from the relation what is the expected $\langle M \rangle$, and calculate $\mu = m - \langle M \rangle$ for this one object. The use of the inverse relation is at first intuitively repugnant because one tends to look at the predictor of M, $(1/a')(p - b')$, similarly as one looks at the direct relation. The direct distance moduli are "individuals," whereas the inverse relation is a kind of collective distance indicator: Measure the average p for the sample and calculate from $\langle m \rangle$ and $\langle p \rangle$ the distance modulus. Restriction to one galaxy, which is so natural with the direct relation, means restricting the value of p to the one observed, which is not allowed with the inverse relation.

From a $m - p$ diagram (Figure 3) showing a "calibrator" (nearby) cluster and a more distant cluster, one can easily explain the secret of the inverse relation. Let us put the calibrator sample at 10 pc, so that m = M. The cluster to be measured is at the unknown distance modulus μ and is cut by the magnitude limit m_{lim}. Glide the calibrator cluster along the m axis by the amount of μ. Then the inverse regression lines are superimposed. This means that the observed average of p at m is $\langle p \rangle_m$ for the second cluster, which is the same as for the calibrator cluster at M $= m - \mu$. From this, it follows that $\langle \mu \rangle_m = m - (\langle p \rangle_m - b')/a'$ and, by averaging over all m, that

$$\mu_{est} = \langle m \rangle - (\langle p \rangle - b')/a'. \tag{14}$$

The (p, M) data form a scattered bivariate distribution, and without further knowledge of the reason for the scatter, one has the freedom, within the limits of what is the application and what is known about the selection of p and M parameters, to use either the direct or the inverse relation. Even if the scatter is not due to errors in p or natural processes that shift p at constant M, Figure 3 shows that one may use the inverse relation if the bivariate distributions of the

calibrator and distant samples are the same. On the other hand, even if there is error in p, one may choose to use the direct relation if the application requires it (Bottinelli et al 1986; Lynden-Bell et al 1988). Naturally, another problem and source of biases is that the bivariate distribution may not fulfill the conditions of gaussianity, which are required in the derivation of the regression lines (Bicknell 1992; Ekholm & Teerikorpi 1997), or the calibrator and distant samples have, for example, different measurement accuracy in magnitude (Teerikorpi 1990; Fouqué et al 1990).

Finally, the inverse relation does not require that its calibrators form a volume-limited sample, which is necessary for the correct calibration of the direct relation. This is also illustrated by Figure 3 because the regression line of the calibrator sample is not changed if a portion m > some m_{lim} is cut away from it.

5.2   *Problems in the Use of the Inverse Relation*

In principle, the inverse TF relation seems to be a good solution to the bias of the second kind, though the scatter in the average distance modulus is larger than for the direct relation in the unbiased plateau ($1/\sigma_{\text{inv}}^2 = 1/\sigma_{\text{dir}}^2 - 1/\sigma_{\text{M}}^2$, where σ_{M} is the dispersion of the general luminosity function). In practice, there are a few more serious problems.

It is essential that there should be no selection according to p, say, working against distant, very broad HI-line galaxies. Also noteworthy is that the calibrator slope for the inverse relation, derived from bright nearby galaxies, is not necessarily the correct slope for distant galaxies. This was shown in a concrete manner in the study of the Virgo cluster by Fouqué et al (1990). If the magnitude or diameter measurements are less accurate for the distant sample than for the calibrators, then the correct slope differs from the calibrator slope. If one ignores this problem, the inverse relation will give distances that are too small or a value of H_{o} that is too high. Theoretically, Teerikorpi (1990) concluded that a solution is to use the slope obtained for the distant sample. However, this requires that the general luminosity function is symmetric around a mean value M_{o}.

The correct slope for the inverse relation is especially important because the aim is to extend measurements at once to large distances, i.e. to extreme values of m and p. A small error in the slope causes large errors at large distances. Of course, it is also important to have the direct relation correct, but in any case, caution is required with regard to the expected Malmquist bias of the second kind, whereas the very absence of the bias is motivation to use the inverse relation.

Hendry & Simmons (1994) made numerical experiments in order to see what is needed of the calibrators in order to produce the correct inverse slope. Adjusting their experiments to the Mathewson et al (1992a,b) data, they concluded that if the number of calibrators is less than 40, the uncertainty $\sigma_{slope} > |direct$

slope − inverse slope|. In other words, with a small calibrator sample, we may think that we use the inverse relation, whereas, in fact, we actually have determined and therefore use the direct one.

Recently, Hendry & Simmons (1994, 1995) formulated the inverse TF distance estimator within the framework and language of mathematical statistics, which confirmed the earlier conclusions on its unbiased nature as regards the Malmquist bias of the second kind. Further discussion on the statistical properties of the inverse TF relation as a distance indicator may be found in Triay et al (1994, 1996) [see also Appendix of Sandage et al (1995) for illuminating notes].

6.   THE BIAS OF THE FIRST KIND IN DIRECT AND INVERSE TF DISTANCE MODULI

In their useful study, Landy & Szalay (1992) initiated the recent discussions on the general or "inhomogeneous" Malmquist bias and its correction, i.e. how to deal with the bias of the first kind in the case of a general space density distribution. True, they encountered some problems in the practical application of the formula derived in their paper. In fact, the basic problem was that they assumed in the beginning a distance indicator that has a zero bias of the second kind (which is unbiased at all distances) and then derived an expression for the bias of the first kind.

In Teerikorpi (1993) the problem of the general correction was discussed with explicit reference to direct and inverse TF relations, which served to clarify some of the points raised by Landy & Szalay (1992). It was noted that Malmquist's formula (Equation 3) was a general one, and in modern terms is best interpreted as applicable to the direct TF relation, for a constant value of p (Malmquist's "star class") and requires that the limiting magnitude $(m_l) = \infty$. In that case, the distribution of distance moduli $N(\mu)$ refers to all moduli that could be observed without any cutoff in the magnitudes.

Interestingly, Feast (1972) had already given a formula quite similar to Equation 3, but now with a quite different meaning for the distribution of $N(\mu)$: In his formulation, $N(\mu)$ must be regarded as the distribution of the derived distance moduli of the galaxies in the observed sample. Inspection of Feast's (1972) derivation reveals the implicit assumption that the bias of the second kind is zero, and the end result was the same as that of Landy & Szalay (1992). As the inverse TF relation distance moduli have the second bias = 0, one can conclude that Feast's and Landy & Szalay's variant of Malmquist's formula applies to the inverse TF distance moduli.

Landy & Szalay's (1992) paper gave rise to a burst of independent discussions. Similar conclusions as in Teerikorpi's (1993) paper on the general

correction and the inverse TF relation were given by Feast himself (1994) and Hendry & Simmons (1994) (see also Hudson 1994 and Strauss & Willick 1995).

To reiterate, Malmquist's formula (Equation 3), with $N(\mu)$ refering to all distance moduli ($m_{lim} = \infty$) in the considered sky direction, applies to the direct TF moduli. In the case of the inverse TF relation, $N(\mu)$ is the observed distribution. Hence, for making corrections of the first kind for the direct distance moduli, one needs information on the true space density distribution of galaxies, and the selection function (magnitude limit) does not enter the problem. Corrections for the inverse moduli depend directly on the distribution of apparent magnitudes (distance moduli) in the sample and, hence, on the selection function. In this sense, the biases of the first and second kind for the direct and inverse distance moduli have curious complementary properties.

One might be content with the above conclusions and try to use the inverse relation in situations where the bias of first kind appears: The correction does not need the knowledge of the true space distribution. In practice, the high inhomogeneity of the local galaxy universe requires corrections for differing directions, which divides galaxy samples into small subsamples where the detailed behavior $N(\mu)$ is difficult to derive.

The general Malmquist bias for the inverse distance moduli, even in the case of a homogeneous space distribution, can be quite complicated. Because the observed $N(\mu)$ usually first increases, reaches a maximum, and then decreases to zero, the bias is first negative (too small distances), then goes through zero, and at large derived distances is positive (i.e. the distances are too long). Something like this is also expected simply because the average bias of the whole sample should be zero.

An important special case of the first Malmquist bias of the direct distance moduli is that of a homogeneous space distribution, which shifts standard candles in a Hubble diagram by a constant amount in magnitude and leaves the expected slope of 0.2 intact. In this manner, Soneira (1979) argued for the local linearity of the Hubble law, in contrast to a nonlinear redshift-distance relationship espoused by Segal, where he argues against debilitating bias effects.

It has been suggested that the distribution of galaxies at least up to some finite distance range could be fractal in nature, with fractal dimension $D \approx 2$ (e.g. Di Nella et al 1996). In that case one might be willing to use, instead of the classical Malmquist correction (for $D = 3$ or homogeneity), Equation 5, which corresponds to the average density law proportional to r^{D-3}. However, though the classical formula applies to every direction in a homogeneous universe, the deformed ($D = 2$) formula is hardly useful in any single direction of a fractal universe because of the strong inhomogeneities. Note however that in this case the argument by Soneira (1979) would be equally valid.

7.   MALMQUIST BIAS AND THE GREAT ATTRACTOR

A good example of the difficulties in establishing the distance scale within the local space is provided by the attempts to determine whether there is a differential infall velocity field around the Great Attractor at about $V_{cosm} \approx 4500$ km/s (Lynden-Bell et al 1988), or alternatively, a bulk flow of galaxies in this same direction of the Hydra-Centaurus supercluster, as earlier proposed by Tammann & Sandage (1985). In principle, with good distances to galaxies in this direction, extending beyond the putative Great Attractor, one would be able to construct a Hubble diagram, velocity vs distance, which in the former case should show both the foreground and backside infall deviations from the linear Hubble law. Such deviations have been detected in the direction of the Virgo cluster (Tully & Shaya 1984; Teerikorpi et al 1992). When aiming at the Great Attractor or the Hydra-Centaurus complex, one must look four to five times farther, hampered by the additional complication of the Zone of Avoidance at low galactic latitudes.

Though different studies agree that there is a flow towards the Great Attractor region, the question of the backside infall remains controversial, after some evidence supporting its detection (Dressler & Faber 1990a,b). Landy & Szalay (1992) suggested that the Malmquist bias (of the first kind) could cause an apparent backside infall signal behind a concentration of galaxies: In the velocity-vs-distance diagram, the distances in front of the concentration, where the density increases, will come out smaller, whereas those calculated behind the concentration, where the density decreases, will be larger on the average than predicted by the classical Malmquist formula.

Mathewson et al (1992a) concluded on the basis of a sample of 1332 southern spirals with TF I-mag distances (Mathewson et al 1992b) that the Hubble diagram does not show any backside infall into the Great Attractor. They thus questioned its very existence. This and Landy & Szalay's (1992) suggestion led Ekholm & Teerikorpi (1994) to investigate the outlook of the velocity-distance and distance-velocity diagrams in the presence of a mass concentration, utilizing a synthetic spherically symmetric supercluster and making comparisons with the data of Mathewson et al (1992a,b). The background method was given by a discussion of four alternative approaches (direct and inverse TF, velocity vs distance, and distance vs velocity) (Teerikorpi 1993). We concluded there that the most promising approach for the detection of the background infall is the analysis of the data in the sense "distance vs velocity." The other way via the classical Hubble diagram needs uncertain Malmquist corrections of the first kind. On the other hand, a p-class analysis (related to MND and TEC) showed that the available sample is not deep enough for the background

infall to be detectable using the direct TF relation (Ekholm & Teerikorpi 1994). However, the inverse TF relation is capable of revealing the backside infall, if it exists. For this, it is crucial to know the relevant inverse slope and to resolve the other problems hampering the inverse relation (see Section 5.2).

Federspiel et al (1994) made an extensive analysis of the Mathewson et al (1992a,b) sample using the Spaenhauer diagrams with their "triple entry corrections" (TEC). They identified clearly the Malmquist bias of the second kind and concluded that the dispersion in the I-mag TF relation is relatively large ($\sigma = 0.4$–0.7 mag), depending on the TF parameter p. After making first order bias corrections, based on an underlying Hubble law, they concluded that there is no detectable backside infall, though there is in the foreground a bulk flow of about 500 km/s, apparently dying out before the putative Great Attractor is reached. A question is whether the first order correction, based on the assumed Hubble law, could lead to a null result or a vicious circle. A demonstration against such a vicious circle was made by Ekholm (1996) using the synthetic supercluster. The Hubble law and the TEC formalism for "fast rotators" should reveal the backside infall, if it exists.

Hudson (1994) calculated inhomogeneous Malmquist corrections to the direct D_n-σ distance moduli in the Mark II sample of E and SO galaxies: He used the density field that he had derived from another, larger sample of galaxies with redshifts known. As a result, Hudson concluded that the apparent backside infall, visible in the Hubble diagram when the homogeneous Malmquist correction was made, was significantly reduced after the new corrections. This is in agreement with the conclusion by da Costa et al (1996).

A special problem is posed by galactic extinction. For instance, the center of the putative Great Attractor, at $l \approx 307°$, b $\approx 9°$, is situated behind the Zone of Avoidance, and the Local Group apex of Rubin et al's (1976) motion observations, at $l \approx 163°$, b $\approx -11°$, was also at low latitudes. Whatever the role of extinction is in those particular cases, in general its influence is subtle and needs special attention. One is not only concerned with good estimates of extinction in the directions of sample galaxies. The variable galactic extinction also creeps in indirect ways into the analyses of galaxy samples. An extinction term must be added into the formula of normalized distances in MND, because galaxies behind an enhanced extinction look fainter in the sky; hence, their "effective" limiting magnitude is brighter (the normalized distance is shorter). In the methods where one uses the limiting magnitude to calculate the importance of the bias of the second kind, one should take into account the extra factor due to extinction (also the inclination effect of internal extinction; Bottinelli et al 1995). Change in the effective limiting magnitude means that other things being equal, the galaxies in the direction of the enhanced extinction are influenced by the second bias already at smaller true distances than are their counterparts in

more transparent parts of the sky. This effect is not allowed for just by making normal corrections to the photometric quantities in the TF relation.

Influence of the extinction is also felt when one uses infrared magnitudes where the individual extinction corrections are small. Such samples are usually based on B-mag or diameter-limited samples that have been influenced by variable B extinction, which changes the effective limiting magnitude in different directions of the sky. Depending on the correlation between B and the magnitude in question, this influence of extinction propagates as a kind of Gould's effect (Section 8.3), into the bias properties of the latter.

8.   SOME RECENT DEVELOPMENTS

In this section I gather together some of the most recent results and ideas concerning the Malmquist biases of the first and second kind.

8.1   *Sosies Galaxies and Partial Incompleteness*

Witasse & Paturel (1997) have discussed the effect of partial incompleteness on the derivation of the Hubble constant, when Kapteyn's Equation 1 is used to make the bias (of the second kind) correction at each redshift. They inspected the completeness of their sample of sosies galaxies ["look-alike"; the distance indicator introduced by Paturel (1984)] and concluded that there is partial incompleteness starting at $B = 12.0$ mag, relative to the assumed $10^{0.6m}$ law. (Another possibility, which Witasse & Paturel (1997) mention for the first time in this kind of work, is that the apparent incompleteness may actually reflect a fractal distribution of galaxies.) Inserting the empirical selection probability into Kapteyn's formula, they could reduce the Hubble constant by about 15%, as compared with the assumption of a complete sample and obtained $H_o \approx 60$ from 181 sosies galaxies. In comparison, Sandage (1996b) applied Paturel's look-alike idea to the sosies galaxies of M31 and M101 and derived by the Spaenhauer method $H_o \approx 50$.

8.2   *About the Unbiased Plateau in the MND*

In the method of normalized distances (MND), the basic concept is the unbiased plateau, which corresponds to the separate unbiased parts of the Spaenhauer diagram in TEC. After its first utilization by Bottinelli et al (1986), several new developments have matured the understanding of the unbiased plateau. Bottinelli et al (1995) recognized that in the Equation 12 of MND, one should add terms describing how the effective magnitude limit changes due to internal extinction (inclination effect) and galactic extinction; it is also possible that the limiting magnitude depends on the TF parameter p (Bottinelli et al 1988a). One must also include the type dependence in the method (Theureau et al 1997).

Numerical simulations by Ekholm (1996) supported the reliability of MND. An interesting result was that in certain kinds of studies (e.g. for determination of the slope of the TF relation), it is admissible to use galaxies somewhat beyond the unbiased plateau, which increases the sample. In fact, it is a handicap in MND, as well as in TEC, that the number of "useful" galaxies remains small in the unbiased regions, e.g. when one determines the value of H_o. One remedy is to use increasingly large samples. For instance, Bottinelli et al (1986) used the total number of galaxies of 395, and the size of the adopted plateau was 41. Theureau et al (1997) used a KLUN sample (with diameters) that was, after necessary restrictions, 4164, and the adopted plateau contained 478 galaxies. It seems that generally the visible empirical plateau contains about 10% of the total sample. Theureau et al (1997) confirmed this by an analytical calculation, where the cumulative error of $\langle \log H \rangle$ was seen to reach about 1% when the fraction of the sample is 0.1.

Because of the small number of plateau galaxies, one would be willing to use for the determination of H_o the inverse relation, where, in principle, one could use the whole sample. However, the various problems with the inverse relation (Section 5) must be solved before it can be safely used as an independent distance indicator.

8.3 *Gould's Effect*

Gould (1993) pointed out a complication present when a sample of galaxies to be used, for example, for infrared I-mag TF relation, is constructed from a sample originally based on selection criteria other than those of I mag, e.g. apparent diameter. The Malmquist bias of the first kind in the distance moduli from the I-mag TF relation does not now generally depend on the squared dispersion $\sigma_I^2 = \langle \in_I^2 \rangle$ of the I-mag TF relation nor on the squared dispersion $\sigma_D^2 = \langle \in_D^2 \rangle$ of the diameter relation, but on the covariance $\langle \in_I \in_D \rangle$ between the corresponding logarithmic distance errors $\in$. An interesting extreme case is when this covariance is zero, i.e. the deflections about the two TF relations are independent. Then there should be no Malmquist bias in the distance moduli from the I-mag TF relation. This is easy to understand: Though the original D-limited sample was selected "from the sky," the second set of I-mag measurements produces symmetrical residuals around the TF relation because of the assumed independence on D residuals and because in this case there is no I-mag limit (cf also Section 3.2 in Landy & Szalay 1992).

In practice, it may be difficult to find such pairs of observables that correlate with a common distance-independent parameter (e.g. TF parameter *p*), but have independent deflections (larger-than-average galaxies tend to be also more luminous than average). Also, it should be noted that the above argument is valid only if one could measure I for all the galaxies first taken from the

D-limited sample, i.e. if the I limit was really $=\infty$. In fact, though to measure the relevant covariance is one approach, Gould (1993) also sees the described problem as supporting the use of the "good old" B-band TF-relation. [For further discussions of Gould's effect, see Willick (1994) and Strauss & Willick (1995).]

In the KLUN project (e.g. Paturel et al 1994), the sample of 5174 spirals has been selected on the basis of apparent size D_{25} (in B), and in the analysis of the diameter TF distance moduli, Gould's effect should not appear. However, in such cases a somewhat related problem is that the measured diameters contain measurement error, and when one constructs a diameter-limited sample from measured galaxies, there is a Malmquist effect due to the dispersion $\sigma_{\in}$ in the measurement error: The sample contains an excess of overestimated apparent diameters. Ekholm & Teerikorpi (1997) pointed out that this may have a significant influence on the results, especially on those from the inverse TF relation where all galaxies (in view of the method's supposedly unbiased nature) are used. Assume now that the apparent sizes of such a diameter-limited sample are once more measured. Because the first and second measurement errors are independent, their influence vanishes from Gould's covariance, and now the second sample has correct measured apparent sizes, on the average. In practice, such a remeasurement of large samples is out of the question, and one has to be aware of the problem.

8.4   *Simulation Approach*

As noted above, Scott (1957) was a pioneer in making numerical simulations, at that time using tables of random numbers in the study of how selection effects influence the distribution function of standard candles and hence the inferred distances. More recently, computers have often been used in this manner, either to show the existence of or to illuminate some effect, for testing a correction method, or for using a Monte Carlo procedure as an integral part of the method. An advantage of such experiments, found in many of the mentioned references, is that one can construct a synthetic galaxy universe where the true distances and values of other relevant parameters are known and where imposition of the known selection effects simulates what the astronomer sees "in the sky."

Simulations have been used to show what happens in the classical Hubble diagram when the line of sight traverses a concentration of galaxies (e.g. Landy & Szalay 1992; Ekholm & Teerikorpi 1994; Section 6.5.2 in Strauss & Willick 1995). However, the most extensive applications have concerned the methods and Malmquist corrections needed when one tries to map the peculiar velocity field, especially using the POTENT algorithm (Dekel et al 1993). Indeed, Dekel et al (1993) made the classical (homogeneous) Malmquist correction to distances inferred from the direct relation, whereas Newsam et al (1995) suggest

the use of the inverse relation in their iterative variant of POTENT. Strauss & Willick (1995) describe their "Method II$^+$," which incorporates random peculiar velocities in a maximum likelihood method, applicable either to the direct or, preferably, to the inverse distance indicator. Nusser & Davis (1995) use the inverse relation in their method for deriving a smoothed estimate of the peculiar velocity field, and they support the method using simulations on a synthetic data set. Freudling et al (1995) gives several examples of how the peculiar velocity field is deformed because of unattended bias, including Gould's effect.

Clearly, theoretical understanding of how the Malmquist biases affect studies of the peculiar velocities is rapidly advancing. On the balance, it is worthwhile to be cautious of how successfully one can apply the actual inverse relation, which is favored in such theoretical studies, to the real data.

9.   CONCLUDING REMARKS

This review concentrates on the basic behavior of the selection biases affecting classical photometric (or diametric) distance indicators, which will always have a central role when large numbers of galaxies are analyzed (morphological luminosity classification, TF, and Faber-Jackson relations). Large samples have the special advantage that one may investigate in detail their properties, completeness, and composition and recognize the relevant selection effects, and hence put the distance indicator on a safe basis.

Along with such major indicators, which are practical for large fundamental samples, there are more specialized methods that complement them, especially within the Local Supercluster, and may provide more accurate distances for a number of individual galaxies. If the distance scale that a "new generation" indicator erects significantly deviates from that obtained from the classical, carefully scrutinized methods, then the first task is to ask whether the new methods have some source of bias. As historically evidenced, distance indicators need sufficient time to mature, and in the long run all the methods, when better understood, should result in consistent distances. Such problems have been suspected in recent years, e.g. in the methods of planetary nebulae luminosity function (PNLF) and of surface brightness fluctuations, which have given systematically smaller distances than Cepheids in galaxies where comparison has been possible (Gouguenheim et al 1996; Tammann et al 1996). Generally, selection effects tend to bias true distances to incorrect smaller values. This rule seems to apply also to the method of PNLF when one attempts to extend it to galaxies in or beyond the Virgo cluster, though it may work very well for close galaxies (Bottinelli et al 1991).

The Hubble Space Telescope has given remarkable building blocks for the distance ladder, especially by the measurement of many new Cepheid distances for galaxies that can be used for calibrating secondary distance indicators (e.g.
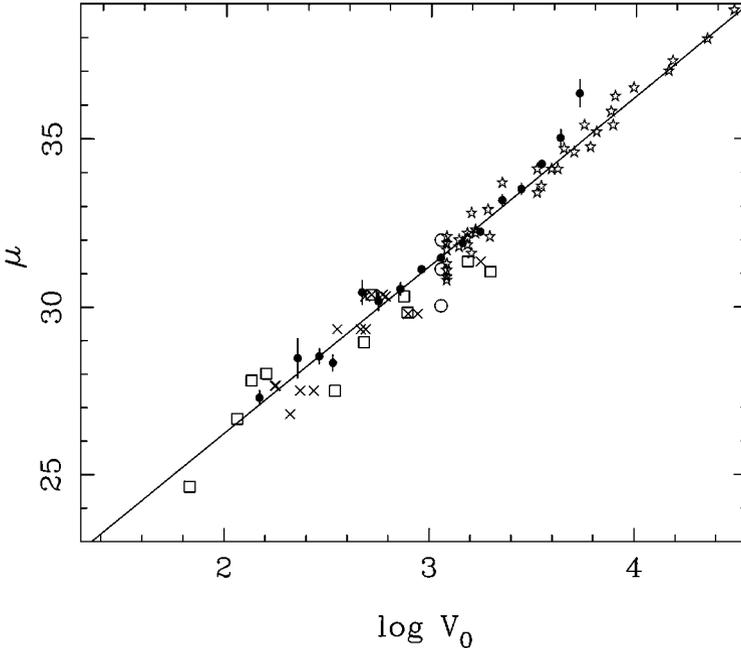
Figure 4   Three distance indicators and the Hubble law are illustrated: SN Ia (*stars*), the B-magnitude TF distance moduli μ from the unbiased plateau of the normalized distance method (*dots*: averages of several galaxies), and Cepheids used for calibrating the supernovae method and the TF relation (*crosses*: individual Cepheid-galaxies, *squares*: groups containing a Cepheid-galaxy). Supernovae data are from Tammann et al (1996) and TF data from Theureau et al (1997). Radial velocity V_o is corrected for the Virgo-centric velocity field. The Hubble line corresponds to $H_o = 56$ km/s/Mpc.

Freedman 1996; Tammann et al 1996). This allows different distance indicators to be based on roughly similar sets of calibrating galaxies and, without intermediaries other than the Hubble law, allows comparison of the resulting distance scales. Differences should then directly reflect problems in one or both of the methods. Presently, there is one pair of distance indicators that can be usefully compared in this way: supernovae of type Ia (SN Ia) (Tammann et al 1996) and the TF relation for D_{25} and B, as applied by Theureau et al (1997) on the KLUN-sample. Figure 4 shows the data, together with single Cepheid galaxies, on the μ-vs-log V_o diagram. The KLUN galaxies have been taken from the unbiased plateau of MND; each symbol refers generally to an average of a few tens of galaxies. The straight line corresponds to the Hubble law with $H_o = 56$ km/s/Mpc, which describes the distribution of individual Cepheid galaxies, groups containing a Cepheid galaxy, TF galaxies, and the SN Ia galaxies in the range of $25 < \mu < 39$. Rather than claiming that this diagram

gives any final distance scale, I show it as an encouraging sign of how different distance indicators begin to give consistent results when special attention is paid to the problem of selection bias. It should be noted that SN Ia, with their small dispersion σ, high luminosity, and detection requirements that are not solely dictated by the maximum luminosity magnitude, are expected to be little affected by the Malmquist bias of the second kind in the distance range shown.

It has been said that "cosmology is a study of selection effects" (Y Baryshev, private communication). This view of what cosmologists are doing may seem a little too dull; nevertheless, there is a deep truth in it, too. It should be an intimate part of the methodology to try to recognize different kinds of selection effects when one attempts to build an unbiased picture of the universe (leaving aside the profound question whether such an enterprise is ultimately at all possible). Often in the forefront of scientific discoveries, one rather likes to ignore selection effects because in order to positively identify these, one should have enough collected data, and this rarely happens in the avant-garde phase. Also, selection effects sometimes produce apparent phenomena that, if true, would certainly be more interesting than the mechanisms of selection and bias that naturally attract less attention.

Another point worthy of emphasis is that the selection biases do not vanish anywhere, even though the astronomical data are accumulated beyond old magnitude limits. Problems are shifted towards larger distances and fainter magnitudes, and new generations of astronomers have to learn how the biases reappear.

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