Scaling Laws in the Distribution of Galaxies

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Research done during the previous century established our Standard Cosmological Model. There are many details still to be filled in, but few would seriously doubt the basic premise. Past surveys have revealed that the large-scale distribution of galaxies in the Universe is far from random: it is highly structured over a vast range of scales. Surveys being currently undertaken and being planned for the next decades will provide a wealth of information about this structure. The ultimate goal must be not only to describe galaxy clustering as it is now, but also to explain how this arose as a consequence of evolutionary processes acting on the initial conditions that we see in the Cosmic Microwave Background anisotropy data.

In order to achieve this we will want to describe cosmic structure quantitatively: we need to build mathematically quantifiable descriptions of structure. Identifying where scaling laws apply and the nature of those scaling laws is an important part of understanding which physical mechanisms have been responsible for the organization of clusters, superclusters of galaxies and the voids between them. Finding where these scaling laws are broken is equally important since this indicates the transition to different underlying physics.

In describing scaling laws we are helped by making analogies with fractals: mathematical constructs that can possess a wide variety of scaling properties. We must beware, however, of saying that the Universe *is* a fractal on some range of scales: it merely exhibits a specific kind of fractal-like behavior on those scales. We exploit the richness of fractal scaling behavior merely as an important supplement to the usual battery of statistical descriptors.

We review the history of how we have learned about the structure of the Universe and present the data and methodologies that are relevant to the question of discovering and understanding any scaling properties that structure may have. The ultimate goal is to have a complete understanding of how that structure emerged. We are getting close!

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I. PHYSICAL COSMOLOGY

With the discovery of the Cosmic Background Radiation by Penzias and Wilson (1965), cosmology became a branch of physics: there was a well defined framework within which to formulate models and confront them with observational data. Prior to that there had been a few important observations and a few important solutions to the Einstein Field Equations for General Relativity. We suspected that these were somehow connected: that the Friedman-Lemaitre solutions of the Einstein field equations described the cosmological redshift law discovered by Hubble.

With the discovery of the background radiation we were left in no doubt that the Universe had a hot singular origin a finite time in our past. That important discovery also showed that our Universe, in the large, was both homogeneous and isotropic, and it also showed the appropriateness of the Friedman-Lemaitre solutions.

The establishment of the "Big Bang" paradigm led to a search for answers, in terms of known physical laws, to key questions: why was the Universe so isotropic, how did the structure we observe originate? and so on. Cosmologists built models involving only known physics and confronted them with the data. Cosmology became a branch of physics with a slight difference: we cannot experiment with the subject of our discussion, the Universe, we can only observe it and model it.

With the current round of cosmic microwave background anisotropy maps we are able to see directly the initial conditions for galaxy formation and for the formation of large-scale structure. That observed structure is thought to reflect directly the fluctuations in the gravitational potential that gave birth to cosmic structure and it is a consequence of the physics of the early universe. The goal is to link those initial conditions with what we see today.

The aim of this article is to show how the "homogeneous and isotropic Universe with a hot singular origin" paradigm has emerged, and to explain how, within this framework, we can quantify and understand the growth of the large scale cosmic structure.

A. Cross-disciplinary physics

Gravitation is the driving force of the cosmos and so Einstein's General Theory of Relativity is the appropriate tool for modelling the Universe. However, that alone is not enough: other branches of physics have played a key role in building what has emerged as a "Standard Model" for cosmology.

Nucleosynthesis played an early role in defining how the light elements formed (Alpher *et al.*, 1948): the abundances of Helium and Deuterium play a vital part in confronting our models with reality. In following how the cosmic medium cooled sufficiently to enable gravitational collapse to form galaxies and stars we need to understand some exotic molecular chemistry.

Today, our understanding of high energy physics plays a key role: some even defined a new discipline and refer to it as "astro-particle physics". We have strong evidence that there is a substantial amount of dark matter in galaxies and clusters of galaxies. So far we have not been able to say what is the nature of this dark matter. There is also growing evidence that the expansion of the Universe is accelerating: this would require an all-pervading component of matter or energy that effectively has negative pressure. If this were true we would have to resurrect Einstein's cosmological constant, or invoke some more politically correct "fifth force" concept such as quintessence.

B. Statistical mechanics

The statistical mechanics of a self-gravitating system is a totally nontrivial subject. Most of the difficulty arises from the fact that gravitation is an always-attractive force of infinite range: there is no analogue to the Debye shielding in plasma physics. Perhaps the most outstanding success was the discovery by Jeans in the 1920's of equilibrium solutions to the Liouville equation for the distribution function of a collection of stars (the Jeans Theorem). This has led to a whole industry in galaxy dynamics, but it has had little or no impact on cosmology where we might like to view the expanding universe with galaxies condensing out as a phase transition in action.

This has not deterred the brave from tackling the statistical mechanics or thermodynamics of self-gravitating systems, but it is perhaps fair to say that so far there have been few outstanding successes. The discussion by Lynden-Bell and Wood (1968) of the so-called gravo-thermal collapse of a stellar system in a box is probably as close as anyone has come. It was only in the 1970's that cosmologists "discovered" the two-point clustering correlation function for the distribution of galaxies and it was not until the late 1980's with the discovery by de Lapparent *et al.* (1986) of remarkable large scale cosmic structure that we even knew what it was we were trying to describe.

The early work of Saslaw (1968, 1969) on "Gravithermodynamics" predated the knowledge of the correlation function. Following the discovery of the correlation function we saw the work of Fall and Severne (1976), Kandrup (1982), and Fry (1984b), providing models for the evolution of the correlation function in various approximations.

One major problem was how to describe this structure. By 1980, it was known that the two-point correlation function looked like a power law on scales[1] $< 10h^{-1}$ Mpc. It was also known that the 3-point function too had a power law behavior and that it was directly related to sums of products of pairs of two-point functions (rather like the Kirkwood approximation). However, N-point correlation functions were not really evocative of the observed structure and were difficult to measure past $N = 4$.

Two suggestions for describing large scale cosmic structure emerged: void probability functions proposed by White (1979) and measured first by Maurogordato and Lachieze-Rey (1987) and multifractal measures (Jones *et al.*, 1988), the latter being largely motivated by the manifest scaling behavior of the lower order correlation functions on scales $< 10h^{-1}$ Mpc. Both of these descriptors encapsulate the behavior of high order correlation functions.

C. Scaling laws in physics

The discovery of scaling laws and symmetries in natural phenomena is a fundamental part of the methodology of physics. This is not new: we can think of Galileo's observations of the oscillations of a pendulum, Kepler's discovery of the equal area law for planetary motion and Newton's inverse square law of gravitation. Some authors claim that the actual discovery of the scaling laws is attributable to Galileo in the context of the strength of materials as discussed in his book *Two New Sciences* (Peterson, 2002).

The establishment of a scaling relationship between physical quantities reveals an underlying driving mechanism. It is the task of Physics to understand and to provide a formalism for that mechanism.

The self-affine Brownian motion is a good example for visual illustration of a scaling process (see Fig. 1). In this case scaling is non-uniform, because different scaling factors have to be applied to each coordinate to keep the same visual appearance.

The breaking of symmetries and of scaling laws is equally important and has played a key role in 20th century physics. Scale invariance is typically broken when some new force or phenomenon comes into play, and the result can look far more significant than it really is. Dubrulle and Graner, 1994; Graner and Dubrulle, 1994 have suggested that this may be the case for the Titius–Bode law (which is, of course, not a law, and can be traced back before Titius and Bode at least to David Gregory in 1702). Their point is that, if the primordial proto- planetary disk had a power-law distribution of density and angular momentum then any process that forms planets will give them something like the Titius–Bode distribution of

[1] The natural unit of length to describe the large scale structure is the megaparsec (Mpc): 1 Mpc = 10^6 pc $\simeq 3.086 \times 10^{22}$ m $\simeq 3.26 \times 10^6$ light years. h is the Hubble constant in units of 100 Mpc^{-1} km s^{-1}.

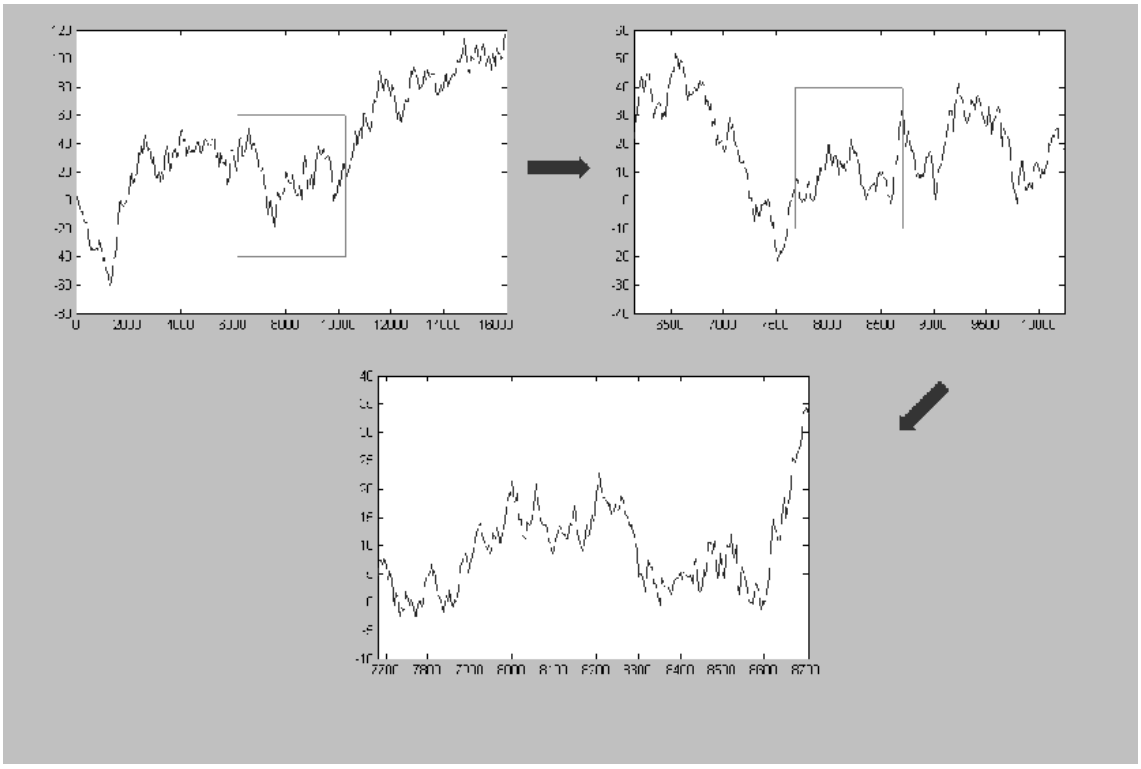

FIG. 1 Scaling relations in one-dimensional Brownian motion $x(t)$. In successive zooms the vertical coordinate (x) is multiplied by 2, while the horizontal coordinate (the time t) is multiplied by 4 to properly rescale the curve.

orbit sizes. Thus the distribution cannot be used as a test for any particular formation mechanism.

Within cosmology, some of the examples of quantized redshifts reported over the years (Burbidge, 1968; Burbidge and Burbidge, 1967; Tifft, 1976) may have been analogous cases, where the "new phenomenon of physics" was observational selection effects resulting when strong emission lines passed into and out of the standard observed wavelength bands.

As we shall see, there are important scaling relationships in the spatial distribution of galaxies. This scaling is almost certainly a consequence of two factors: the nature of the initial conditions for cosmic structure formation and the fact that the gravitational force law is itself scale-free.

This scaling is observed to break down at very large distance. This breakdown is a consequence of the large-scale homogeneity of the Universe and of the fact that the Universe has a finite age: gravitational agglomeration of matter has only been able to spread over a limited domain of scales, leaving the largest scales unaffected.

The scaling is also expected to break down for small objects where non-gravitational forces have played a role: gas-dynamic processes play an important role in the later stages of galaxy formation. There are important scaling relationships among the properties of galaxies which provide clues to the mechanisms of their formation. We do not deal with these in detail here, although the main scaling laws in the galaxy properties are summarized in Sect. VII.A.5.

D. Some psychological issues

Cosmology presents physics with a formidable challenge. The Universe is not a bounded and isolated system. The Universe is far from being in any form of dynamical equilibrium. The gravitational force is of infinite range and always attractive. Nor can we experiment on the subject of interest, we are mere observers. Thus the usual concepts from statistical physics cannot be simply imported, they have to be redefined to suit these special circumstances.

This process of redefinition is apt to misdirect the struggle for understanding the issues involved and is inevitably frustrating to those who work in statistical physics or who seek to use techniques from statistical physics. Indeed there have been occasions where the notions of the standard model have been abandoned simply in order to exploit standard concepts that would otherwise be invalid (eg.: model universes having one spatial dimension or model universes that have zero mean density in the large). Those papers may be interesting, but they have little or nothing to do with the Universe as we know it.

II. THE COSMIC SETTING

The establishment of a definitive cosmological picture has been one of the triumphs of 20th Century physics. From Einstein's first investigations into relativistic cosmological models, through Hubble's discovery of the cosmic expansion, to

the discovery of the Cosmic Microwave Background Radiation in 1965, most physicists would now agree on the basic ingredients of what might as well be called "the Standard Cosmological Model". The astrophysics of the 21st century will consist largely of filling in and understanding the details of this model: a nontrivial process that will consume substantial human, technical and financial resources.

While there are suggestions that the standard model may not be complete, the data as a whole do not as yet demand any further parametrization such as "quintessence". Of course, as our understanding of fundamental physics deepens, the standard model might be recast in a new wider, more profound, framework such as that offered by brane cosmologies.

A. Key factors

There are several important factors to support our current view of cosmic structure formation:

- The discovery by Hubble in 1928 of the linear velocity-distance relationship for galaxies (Hubble, 1929). This relationship was soon interpreted by Robertson (1928) as being due to the expansion of the Universe in the manner described by the Friedman-Lemaitre cosmological solutions of the Einstein Field equations for gravitation. These solutions described a homogeneous and isotropic Universe emerging from a singular state of infinite density: the Big Bang. Later on, Bondi and Gold (1948) and Hoyle (1948) provided an alternative homogeneous and isotropic expanding model that avoided the initial singularity: the Steady State Theory.
- The discovery in 1965 of the Cosmic Microwave Background Radiation tells us the cosmological framework within which we have to work. Our Universe is, in the large, homogeneous and isotropic; it was initially hot enough to synthesize the element Helium. This is the Hot Big Bang theory promoted early on by Gamow. This discovery signaled the end of the Steady State Theory.
- The observation in 1992 by the COBE satellite of the large-scale structure of the Universe at very early times provides us with precise information about the initial conditions for structure formation. This is ongoing research that will lead to detailed knowledge of the fundamental parameters of our Standard Model and to detailed knowledge of the initial conditions in the Big Bang that resulted in the currently observed structure.

We know a great deal about our Universe. Studies of cosmic structure must fall within the precepts set by our Standard Model or they will simply be dismissed at best as being academic curiosities or at worst as being totally irrelevant.

B. Some caveats

The most important caveat in all of this is the fact that when studying cosmic structure we observe only the luminous constituents of the Universe. It is true that we can observe cosmic structure over an enormous range of the electromagnetic spectrum, but nevertheless we face the prospect that about 85% of what there is out there may forever remain invisible except indirectly though its gravitational influence.

Fortunately, we can directly study the gravitational influence of the dark component in a number of ways. If it is uniformly distributed it has an influence on the overall cosmic expansion and on the physics of the early Universe. We can detect its influence by studying the cosmic expansion law, or by studying the nature of the spatial inhomogeneities seen in the cosmic microwave background radiation. If it is not uniformly distributed it will influence the dynamics of the large scale structure as seen in the velocity maps for large samples of galaxies and it may reveal itself through studies of gravitational lensing.

Our numerical simulations of the evolution of structure can in principle take account of several forms of matter. While this has been a successful program, the lack of detailed knowledge about the nature of the dark matter is nevertheless a serious impediment. Some astrophysicists would turn the problem around and argue that those simulations that best reproduce what is seen will provide important information about the nature of the dark matter.

III. EARLY IDEAS ABOUT THE GALAXY DISTRIBUTION

A. Cosmogony

In the 4th. Century BCE, Epicurus taught that there are an infinite number of worlds like (and unlike) ours, while Aristotle taught that there is only one. Neither hypothesis can currently be falsified, and indeed we may see the continuation of this metaphysical battle in the so-called inflationary cosmological models.

Philosophers since Anaximander (Kahn, 1994) have long debated the true nature of the Universe, presenting often remarkably prescient ideas notwithstanding the lack of any real data. Given the lack of data, the only basis for constructing a Universe was symmetry and simplicity or some more profound cosmological principle.

The ancients saw nested crystalline spheres fitting neatly into one another: this was a part of the then culture of thinking of mathematics (i.e. geometry in those days) as being somehow a fundamental part of nature [2]. Later thinkers such as Swedenborg, Kant and Descartes envisioned hierarchies of nested whirls. While these ideas generally exploited the scientific trends and notions of their time, none of them were formulated in terms of physics. Many are reviewed in Jones (1976) where detailed references to the classical works are given.

[2] Einstein's great intellectual coup was to geometrize the force of gravity: we are governed on large scales by the geometry of space-time manifesting itself as the force of gravity.

Perhaps the first detailed presentation of cosmogonic ideas in the modern vein was due to Poincaré in his *Leçons sur les Hypothèses Cosmogoniques* (Poincaré, 1894), some of which was to be echoed by Jeans in his texts on Astronomy and Cosmogony (Jeans, 1928). Jeans' work is said to have had a profound effect on Hubble's own thoughts about galaxy evolution and structure formation (Christianson, 1995).

B. Galaxies as "Island Universes"

Once upon a time there was a single galaxy. William and Caroline Herschel had drawn a map of the Galaxy (Herschel, 1785) on the basis that the Sun was near the center of the Galaxy, and this image persisted into the 20th Century with the "Kapteyn Universe" (Kapteyn, 1922) which depicted the the Milky Way as a relatively small flattened ellipsoidal system with the Sun at its center, surrounded by a halo of globular clusters. Trumpler (1930) recognized the role played by interstellar absorption; he provided a far larger view of the Galaxy and moved the Sun outwards from the center of the Galaxy to a position some 30,000 light years from the Galactic Center.

Competing with this view was the hypothesis of Island Universes, though at least some astronomers 100 years ago thought that had been completely ruled out. Remember that 100 years ago it was not known that the "nebulae" were extragalactic systems: they were thought of as whirlpools in the interstellar medium.

The controversy between the Great Galaxy and Island Universe views culminated in the great debate between Curtis and Shapley in 1920 (Hoskin, 1976). Shapley, who had earlier placed our Sun in the outer reaches of the Greater Galaxy by observing the distribution of globular clusters[3], defended the Great Galaxy hypothesis and won the day for all the wrong reasons.

However, it was left to Edwin P. Hubble to settle the issue in favour of the Island Universes when he found Cepheid variables in the galaxy NGC6822 and the Andromeda nebula (Hubble, 1925a,b).

There was one anomaly that persisted into the early 1950's: our Galaxy seemed to be the largest in the Universe. This was resolved by Baade who recognized that there were in fact two populations of Cepheid variables (Baade, 1956). This doubled the distances to the external galaxies, thereby solving the problem.

For Hubble and most of his contemporaries what had been found were "field galaxies" largely isolated from one another. This was in part due to the sorts of telescope and their fields of view that Hubble was using (Hubble, 1934, 1936) and also in part due to the lingering effects of the phrase "Island universe" which evoked images of isolation. Indeed, as late as the 1960's, astronomers who should have known better said that galaxies were the building blocks of the Universe (eg: McCrea (1964) and Abell in undergraduate lectures at UCLA 1961-1963).

In fact, most galaxies are clustered. This is implicit in images taken with smaller telescopes having larger fields (Shapley often said that large telescopes were over-rated (Shapley, 1932), perhaps in part because he had deliberately cut himself off from them by moving to Harvard) and explicit in the remarks of Zwicky (1938, 1952) who had begun to look at the Universe through Schmidt-coloured glasses. (The 18" Schmidt telescope on Palomar Mountain came into use a couple of years before).

C. Earliest impressions on galaxy clustering

In the 19th century William Herschel and Charles Messier noted that the amorphous objects they referred to as "nebulae" were more common in some parts of the sky than others and in particular in the constellation of Virgo.

However, clusters of galaxies were not described in detail until the work of Wolf (1924) who described the Virgo and Coma clusters of galaxies. It was not known at that time that the nebulae, as they were then called, were in fact extragalactic systems of stars comparable with our own Galaxy.

Hubble, using the largest telescopes, noted the remarkable overall homogeneity and isotropy of the distribution of galaxies. The first systematic surveys of the galaxy distribution were undertaken by Shapley and his collaborators (often uncited and under-acknowledged wealthy Bostonian women). This lead to the discovery of numerous galaxy clusters and even groups of galaxy clusters.

D. Hierarchical models

The clustering together of stars, galaxies, and clusters of galaxies in successively ordered assemblies is normally called a hierarchy, in a slightly different sense of the dictionary meaning in which there is a one-way power structure. The technically correct term for the structured universes of Kant and Lambert is multilevel. A complete multilevel universe has three consequences. One is the removal of Olbers paradox (the motivation of John Herschel and Richard Proctor in the 19th century). The second, recognized by Kant and Lambert, is that the universe retains a primary center and is therefore nonuniform on the largest cosmic scales. The third, recognized by the Irish physicist Fournier d'Albe and the Swedish astronomer Carl Charlier early in the 20th century is that the total amount of matter is much less than in a uniform universe with the same local density. D'Albe put forward the curious additional notion that the visible universe is only one of a series of universes nested inside each other like Chinese boxes. This is not the same as multiple 4-dimensional universes in higher dimensional space and does not seem to be a forerunner of any modern picture.

[3] We should recall that at about this time Linblad (1926) and Oort (1928) showed that the stars in the Galaxy were orbiting about a distant center, thus clearly placing the Sun elsewhere than at the center.

1. Charlier's Hierarchy

The idea that there should be structure on all scales up to that of the Universe as a whole goes back to Lambert (1761) who was trying to solve the puzzle of the dark night sky that is commonly called "Olber's paradox". (It was not formulated by Olbers and it is a riddle rather than a paradox (Harrison, 1987)). Simply put: if the Universe were infinite and uniformly populated with stars, every line of sight from Earth would eventually meet the surface of a star and the sky would therefore be bright. The idea probably originated with John Herschel in a review of Humboldt's Kosmos where the clustering hierarchy is suggested as a solution to Olber's Paradox as an alternative to dust absorption.

At the start if the 20th century, The Swedish astronomer Carl Charlier provided a cosmological model in which the galaxies were distributed throughout the Universe in a clustering hierarchy (Charlier, 1908, 1922). His motivation was to provide a resolution for Olber's Paradox. Charlier showed that replacing the premise of uniformity with a clustering hierarchy would solve the problem provided the hierarchy had an infinite number of levels (see Fig. 2).

Charlier's idea was not new, though he was the first person to provide a correct mathematical demonstration that Olber's Paradox could indeed be resolved in this way. It should be recalled that he was working at a time before any galaxies had measured redshifts and long before the cosmic expansion was known.

It is interesting that the Charlier model had de Vaucouleurs as one of its long standing supporters (de Vaucouleurs, 1970).

More recently still there have been a number of attempts to re-incarnate such a universal hierarchy in terms of fractal models. Fractal models were first proposed by Fournier d'Albe (1907) and subsequently championed by Mandelbrot (1982) and Pietronero (1987). Several attempts have been made to construct hierarchical cosmological models (a Newtonian solution was found by Wertz (1971), general-relativistic solutions were proposed by Bonnor (1972); Ribeiro (1992); Wesson (1978)). All these solutions are, naturally, inhomogeneous with preferred position(s) for the observer(s), and thus unsatisfactory. So the present trend to conciliate fractal models with cosmology is to use the measure of last resort, and to assume that although the matter distribution in the universe is homogeneous on large scales, the galaxy distribution can be contrived to be fractal (Ribeiro, 2001). Numerical models of deep samples contradict this assumption.

2. Carpenter's law

Edwin F. Carpenter spent his early days at Steward Observatory (of which he was director for more than 20 years, from 1938) scanning zone plates to pick out extragalactic nebulae for later study. In 1931, he found a new cluster in the direction of Cancer (independently discovered by Hubble at about the same time.) He measured its size on the sky, estimated its distance, and counted the number of galaxies, N, he could recognize within its confines. This gave him a sample of 7 clusters with similar data, all from Mt. Wilson plates (5 in the Mt. Wilson director's report for 1929-30 and one then just found by Lundmark). He was inspired to graph $\log(N)$ vs. the linear sizes of the clusters (Carpenter, 1931) and found a straight line relation, that is, a power law in $N(\text{diameter})$, nowhere near as steep as $N \sim D^3$ or N proportional to volume. The then known globular cluster system of the Milky Way (with about 35 clusters within 10^5 pc) also fit right on his curve.

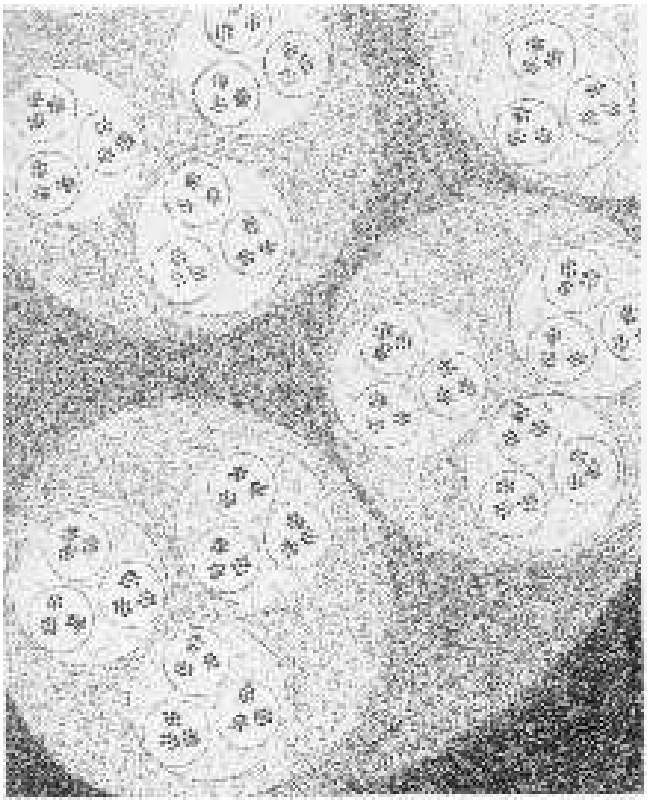

FIG. 2 Hierarchical universes were very popular at the end of the 19th century and the first half of the 20th century. Reproduced from Harrison (2000), Cosmology, Cambridge University Press.

Carpenter later considered a larger sample of clusters and found that a similar curve then acted as an upper envelope to the data (Carpenter, 1938). If his numbers are transformed to the distance scale with $H_0 = 100$ km s^{-1} Mpc^{-1}, then the relations are (de Vaucouleurs, 1971)

$$\log N(\text{max}) = 2.5 + 1.5 \log R(\text{Mpc}) \qquad (1)$$

or

$$\log N(\text{max}) = 2.19 + 0.5 \log V(\text{Mpc}^3) \qquad (2)$$

and the maximum number density in galaxies per Mpc^3 is also proportional to $0.5 \log(V)$. De Vaucouleurs called this Carpenter's law, though the discoverer himself had been somewhat more tentative, suggesting that this sort of distribution (which we would call scale free, though he did not) might mean that there was no fundamental difference among groups,

clusters, and superclusters of galaxies, but merely a non-random, non-uniform distribution, which might contain some information about the responsible process. It is, with hindsight, not surprising that the first few clusters that Carpenter (1931) knew about were the densest sort, which define the upper envelope of the larger set (Carpenter, 1938). The ideas of a number of other proponents, both observers and theorists, on scale-free clustering and hierarchical structure are presented (none too sympathetically) in Chapter 2 of Peebles (1980).

3. De Vaucouleurs hierarchical model

De Vaucouleurs first appears on the cosmological stage doubting what was then the only evidence for galaxy evolution with epoch, the Stebbins-Whitford effect, which he attributed to observational error (de Vaucouleurs, 1948). He was essentially right about this, but widely ignored. He was at other times a supporter of the cosmological constant (when it was not popular) and a strong exponent of a hierarchical universe, in which the largest structures we see would always have a size comparable with the reach of the deepest surveys (de Vaucouleurs, 1960, 1970, 1971). He pointed out that estimates of the age of the universe and of the sizes of the largest objects in it had increased monotonically (and perhaps as a sort of power law) with time since about 1600, while the densities of various entities vs. size could all be plotted as another power law,

$$\rho(r) \sim r^{-x}, \text{with } x \text{ between 1.5 and 1.9.} \qquad (3)$$

By putting "Carpenter's Law" into modern units, de Vaucouleurs showed that it described this same sort of scale-free universe. A slightly more complex law, with oscillations around a mean, falling line in a plot of density vs. size (see Fig. 3), could have galaxies, binaries, groups, clusters, and superclusters as distinct physical entities, without violating his main point that what you see is what you are able to see.

De Vaucouleurs said that it would be quite remarkable if, just at the moment he was writing, centuries of change in the best estimate for the age and density of the universe should stop their precipitous respective rise and fall and suddenly level off at correct, cosmic values. Thus he seemed to be predicting that evidence for a universe older than 10-20 Gyr and for structures larger than 100 Mpc should soon appear. (He held firmly to a value of H_0 near 100 km s^{-1} Mpc^{-1} for most of his later career, except for the 1960 paper where it was 75, but thought of local measurements of H_0 as being relevant only locally).

Remarkable, but apparently true. Instead of taking off again, estimates of the age of the universe made since 1970 from radioactive decay of unstable nuclides, from the evolution of the oldest stars, and from the value of the Hubble constant, increasingly concur. And galaxy surveys have now penetrated a factor 10 deeper in space than the Shane-Wirtanen and Harvard counts in which de Vaucouleurs saw his superclusters.

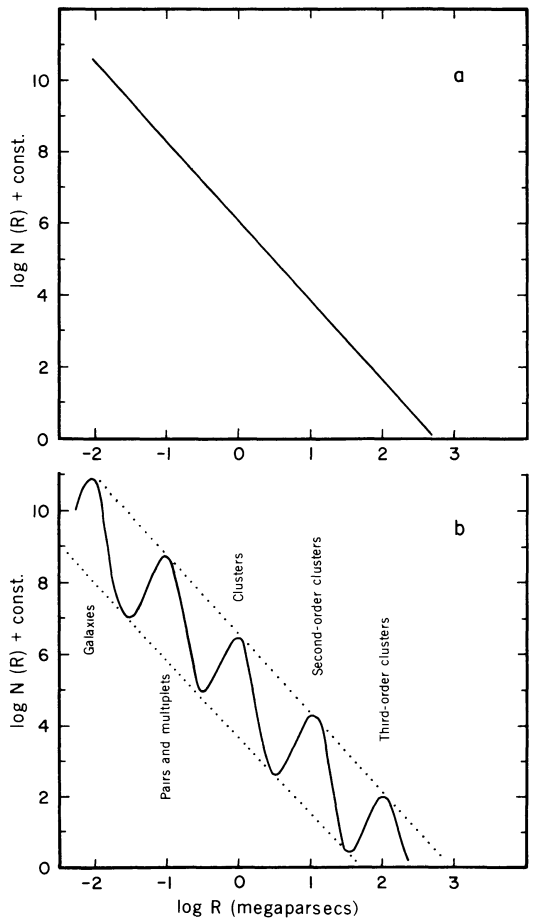

FIG. 3 In this idealized diagram de Vaucouleurs shows two hierarchical frequency distributions of the number of clumps per unit volume. In the top panel there are no characteristic scales in the distribution. This is the model proposed by Kiang and Saslaw (1969). The bottom panel shows a more sophisticated alternative in which the overall decrease of the number of clumps per unit volume does not behave monotonically with the scale, but it displays a series of local maxima corresponding to the characteristic scales of different cosmic structures: galaxies, groups, clusters, superclusters, etc. Reproduced from de Vaucouleurs (1971), Astronomical Society of the Pacific.

E. The cosmological principle

The notion that the Earth is not at the center of the Universe is generally referred to as the "Copernican Principle", though it traces its origins back to Aristarchus who thought that the Sun and the stars were in fact fixed, with the stars being at great distances.

The modern notion that the Universe on the very largest scales should be homogeneous and isotropic appears to have originated with Einstein (1917). At that time there could have been no observational basis for this assumption. However, homogeneity is a consequence of the notion that we are not in a special place in the Universe and the assumptions of homogeneity and isotropy provide for easy solutions of the Einstein field equations. The first cosmological models of Einstein

and of de Sitter were based on this principle. Robertson and Walker derived their famous solution of the Einstein equations using only that principle.

It was frequently stated in the years that followed that the Universe in the large looked homogeneous and isotropic. The first systematic study was Hubble (1926) who used a sample of 400 galaxies with magnitudes, the sample was thought to be complete to magnitude 12.5. He found his counts fitted the relationship

$$\log N(< m) = 0.6m + \text{constant} \tag{4}$$

and concluded, importantly, that "The agreement between observed and computed $\log N$ over a range of more than 8 mag. is consistent with the double assumption of uniform luminosity and uniform distribution or, more generally, indicates that the density function is independent of the distance." He goes on to look at systematics in the residuals in this plot and concludes that they may be due to "... clustering of nebulae in the vicinity of the galactic system. The cluster in Virgo alone accounts for an appreciable part."

Hubble only had data to magnitude 12. Anyone looking at the considerably fainter Shane and Wirtanen's isoplethic maps of galaxy counts based on the Lick Sky Survey (Shane and Wirtanen (1967)), or the more recent Center for Astrophysics (CfA-II) slices data (Geller and Huchra, 1989) might be forgiven for questioning the homogeneity conjecture!

The first demonstration of homogeneity in the galaxy distribution was probably the observation by Peebles that the (projected) two-point correlation function estimated from diverse catalogs probing the galaxy distribution to different depths followed a scaling law that was consistent with homogeneity. The advent of automated plate-measuring machines provided deeper and more reliable samples with which to confirm the uniform distribution number-magnitude relationship. However, at the faintest magnitude levels, these counts show significant systematic deviations from what is expected from a uniform distribution: these deviations are due to the effects of galaxy evolution at early times and their interpretation depends on models for the evolution of stellar populations in galaxies. Recent, very deep studies (Metcalfe *et al.* (2001)) show convincingly "... that space density of galaxies may not have changed much between $z = 0$ and $z = 3$".

The first incontrovertible proof of cosmic isotropy came only as recently as early 1990s from the COBE satellite all-sky map of the cosmic microwave background radiation (Smoot *et al.*, 1992). This map is isotropic to a high degree, with relative intensity fluctuations only at the level of 10^{-5}. With this observation, and with the reasonable hypothesis that the Universe looks the same to all observers (the Copernican Principle) we can deduce that the Universe must be locally Friedman-Robertson Walker, ie: homogeneous as well as isotropic (Ehlers *et al.*, 1968).

IV. DISCOVERING COSMIC STRUCTURE

A. Early catalog builders

Observational cosmology, like most other physical sciences, is technology driven. With each new generation of telescope and with each improvement in the photographic process, astronomers probed further into the Universe, cataloguing its contents.

Early on, Edward Fath used the Mount Wilson 60" telescope to photograph Kapteyn's selected areas. That survey showed inhomogeneities that were later analyzed by Bok (1934) and Mowbray (1938) who demonstrated statistically, using counts in cells, that the galaxy distribution was nonuniform. About this time, Carpenter (1938) noticed that small objects tend to be dense while vast objects tend to be tenuous. He plotted a remarkable relationship between scale and density ranging all the way from the Universe, through galaxies and stellar systems to planets and rock, as it has been explained in Sect. III.D.2. This was perhaps the first example of a scaling relationship in cosmology.

By 1930, the Shapley/Ames catalog of galaxies revealed the Virgo cluster as the dominant feature in the distribution of bright galaxies. It was already clear from that catalog that the Virgo Cluster was part of an extended and rather flattened supercluster. This notion was hardly discussed except by de Vaucouleurs who thought that this was indeed a coherent structure whose flattening was due to rotation.

The Lick Survey of the sky provided extensive plate material that was later to prove one of the key data sets for studies of galaxy clustering. The early isoplethic maps drawn by Shane and Wirtanen (1954) provided the first cartographic view of cosmic structure. Their counts of galaxies in cells was to provide Rubin (1954) and Limber (1954) with the stimulus to introduce the two point clustering function as a descriptor of cosmic structure.

But it was the Palomar Sky Survey using the new 48" Schmidt telescope that was to provide the key impetus in understanding the clustering of galaxies. Zwicky and his collaborators at Caltech systematically cataloged the position and brightness of thousands of brighter galaxies on these plates, creating what has become known as the "Zwicky Catalog". Abell (1958) made a systematic survey for rich clusters of galaxies and drew up a catalog listing thousands of clusters. This has become simply known as the "Abell catalog". Fig. 4 shows a modern image of the cluster Abell 1689 obtained by the ACS camera aboard of the Hubble Space Telescope (HST). A catalog of galaxy redshifts noting the clusters to which galaxies belonged was published in 1956 by Humason *et al.* (1956).

1. The Lick survey

The first map of the sky revealing widespread clustering and super-clustering of galaxies came from the Lick survey of galaxies undertaken by Shane and Wirtanen (1967) using large field plates from the Lick Observatory. This was, or

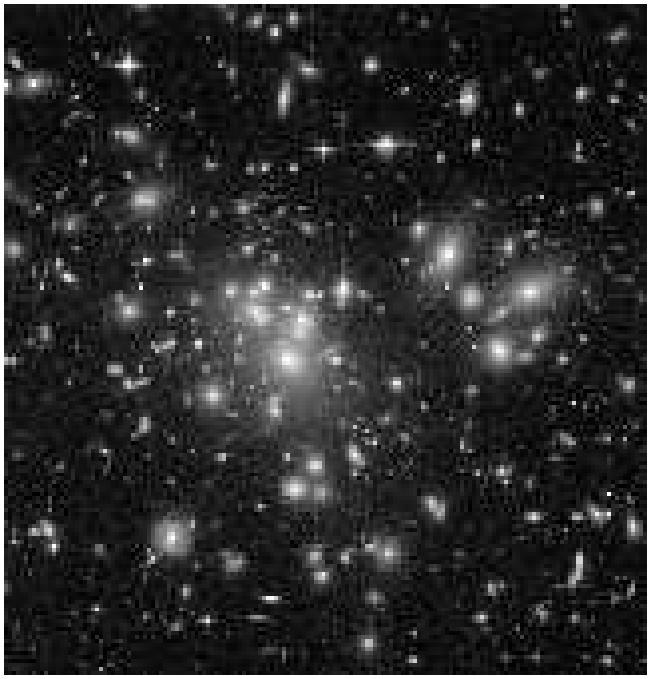

FIG. 4 The cluster of galaxies Abell 1689 at redshift $z = 0.18$ seen by the HST with its recently installed Advanced Camera for Surveys (ACS). The arcs observed amongst hundreds of galaxies conforming the cluster are multiple images of far-away individual galaxies whose light has been amplified and distorted by the total cluster mass (visible and dark) acting as a huge gravitational lens, (image courtesy of NASA, N. Benitez (JHU), T. Broadhurst (The Hebrew University), H. Ford (JHU), M. Clampin (STScI), G. Hartig (STScI), G. Illingworth (UCO/Lick Observatory), and the ACS Science Team, and ESA).

anyhow should have been, the definitive database. It was the subject of statistical analysis by Neyman *et al.* (1953), which was a major starting point for what have subsequently become known as Neyman–Scott processes in the statistics literature. Ironically, although these processes have become a discipline in their own right, they have since that time played only a minor role in astronomy.

Scott in the IAU Symposium 15 (Scott, 1962) mentions that there are clearly larger structures to be seen in these counts, as Shane and Wirtanen (1954) had already noted. They spoke of "larger aggregations" or "clouds" as being rather general features. The Lick survey was later to play an important role in Peebles' systematic assault on the problem of galaxy clustering. Peebles obtained from Shane the notes containing the original counts in 10'x10' cells and computerized them for his analysis. The counts in 1 degree cells had been used first by Vera Cooper-Rubin (as Vera Rubin was then known) to study galaxy clustering in terms of correlation functions, a task set by her adviser George Gamow. Rubin did this at a time when there were no computers. It was Totsuji and Kihara (1969) who first did this on a computer and published the first two-point correlation function as we now know it with the power law that has dominated much of cosmology for the past three decades and more[4].

2. Palomar Observatory sky survey

The two main catalogs of clusters derived from the Palomar Observatory Sky Survey (POSS) were that of Abell (1958) and that of Zwicky and his collaborators (Zwicky *et al.*, 1961–1968).

Abell went on immediately to say that there was significant higher order clustering in his data, giving, in 1958, a scale for superclustering of 24 $(H_0/180)^{-1}$ Mpc. In 1961 at a meeting held in connection with the Berkeley IAU Abell published (Abell, 1961) a list of these "super-clusters", dropped the Hubble constant to 75 km s^{-1} Mpc^{-1} and estimated masses of $10^{16} - 10^{17}\ M_\odot$ with velocity dispersions in the range 1000-3000 km s^{-1}. At about the same time, van den Bergh (1961) remarks that Abell's most distant clusters (distance class 6 having redshifts typically around 50,000 km s^{-1}) show structure on the sky on a scale of some 20°, corresponding to 100 Mpc, for his $H_0 = 180$ km s^{-1} Mpc^{-1}, or about 300 Mpc using current values.

Zwicky explicitly and repeatedly denied the existence of higher order structure (Zwicky and Berger, 1965; Zwicky and Karpowicz, 1966; Zwicky and Rudnicki, 1963, 1966). Some of his "clusters" were on the order of 80 Mpc across (for H_0 less than 100), had significant substructure, and would to any other person have looked like superclusters! Herzog, one of Zwicky's collaborators in the cluster catalog, found large aggregates of clusters in the catalog and had the temerity to say so publicly in a Caltech astronomy colloquium. He was offered "political asylum" at UCLA by George Abell. Karachentsev (1966) also reported finding large aggregates in the Zwicky catalog.

3. Analysis of POSS clusters

Up until about 1960 most of those involved seemed to envisage a definite hierarchy of structures: galaxies (perhaps binaries and small groups), clusters and superclusters. Kiang remarked that the existing data were best described by continuous, "indefinite", clustering: quite different from the clustering hierarchy as understood at the time (Kiang, 1961; Kiang and Saslaw, 1969). Kiang, incidentally, bridged a critical era in data processing, using "computers" (i.e., poorly paid non-PhD labour, mostly women after the style of Shapley) and later on real computers (Atlas). Flin *et al.* (1974) came independently to the same conclusion, and in his presentation at IAU Symposium 63 was scolded by Kiang for not having read the literature.

[4] BJ "discovered" this paper at the time of writing his Review of Modern Physics article (Jones, 1976) while perusing the Publications of the Astronomical Society of Japan in the Institute of Theoretical Astronomy Library in Cambridge. There do not appear to be any citations prior to that time.

The later investigation by Peebles and Hauser (1974) using the power spectrum of the cluster distribution showed superclustering quite conclusively: clusters of galaxies are not randomly distributed and as they are correlated they are themselves clustered. Later analyses revealed a variation of cluster clustering with cluster richness.

Nevertheless, there still remained mysteries to be cleared up: the level measured for clustering of clusters was far in excess of what would be expected on the basis of the measured clustering of the galaxies from which they are built. Many solutions have been proposed to explain this anomaly, including the argument that the Abell catalog is too subjective and biased. However, the phenomenon still persists in cluster catalogs constructed by machine scans of photographic plates.

B. Redshift Surveys

1. Why do this?

Those early catalogs simply listed objects as they appeared projected onto the celestial sphere. The only indication of depth or distance came from brightness and/or size. These catalogs were, moreover, subject to human selection effects and these might vary depending on which human did the work, or even what time of the day it was.

What characterizes more recent surveys is the ability to scan photographic plates digitally (eg: the Cambridge Automatic Plate Machine, APM), or to create the survey in digital format (eg: IRAS, Sloan Survey and so on). Moreover, it is now far easier to obtain radial velocities (redshifts) for large numbers of objects in these catalogs.

Having said that, it should be noted that handling the data from these super-catalogs requires teams of dozens of astronomers doing little else. Automation of the data gathering does little to help with the data analysis!

Galaxy redshift surveys occupy a major part of the total effort and resources spent in cosmology research. Giving away hundreds of nights of telescope time for a survey, or even constructing purpose built telescopes is no light endeavour. We have to know beforehand why we are doing this, how we are going to handle and analyze the data and, most importantly, what we want to get out of it. The early work, modest as it was by comparison with the giant surveys being currently undertaken, has served to define the methods and goals for the future, and in particular have served to highlight potential problems in the data analysis.

We have come a long way from using surveys just to determine a two-point correlation function and wonder at what a fantastic straight line it is. What is probably not appreciated by those who say we have got it all wrong (eg: Sylos Labini *et al.* (1998)) is how much effort has gone into getting and understanding these results by a large army of people. This effort has come under intense scrutiny from other groups: that is the importance of making public the data and the techniques by which they were analyzed. The analysis of redshift data is now a highly sophisticated process leaving little room for uncertainty in the methodology: we do not simply count pairs of galaxies in some volume, normalise and plot a graph!

The prime goals of redshift surveys are to map the Universe in both physical and velocity space (particularly the deviation from uniform Hubble expansion) with a view to understanding the clustering and the dynamics. From this we can infer things about the distribution of gravitating matter and the luminosity, and we can say how they are related. This is also important when determining the global cosmological density parameters from galaxy dynamics: we are now able to measure directly the biases that arise from the fact that mass and light do not have the same distribution.

Mapping the universe in this way will provide information about how structured the Universe is now and at relatively modest redshifts. Through the cosmic microwave background radiation we have a direct view of the initial conditions that led to this structure, initial conditions that can serve as the starting point for N-body simulations. If we can put the two together we will have a pretty complete picture of our Universe and how it came to be the way it is.

Note, however, that this approach is purely experimental. We measure the properties of a large sample of galaxies, we understand the way to analyse this through N-body models, and on that basis we extract the data we want. The purist might say that there is no understanding that has grown out of this. This brings to mind the comment made by the mathematician Russell Graham in relation to computer proofs of mathematical theorems: he might ask the all-knowing computer whether the Riemann hypothesis (the last great unsolved problem of mathematics) is true. It would be immensely discouraging if the computer were to answer "Yes, it is true, but you will not be able to understand the proof". We would know that something is true without benefiting from the experience gained from proving it. This is to be compared with Andrew Wiles' proof of the Fermat Conjecture (Wiles, 1995) which was merely a corollary of some far more important issues he had discovered on his way: through proving the fundamental Taniyama-Shimura conjecture we can now relate elliptic curves and modular forms (Horgan, 1993).

We may feel the same way about running parameter-adjusted computer models of the Universe. Ultimately, we need to understand why these parameters take on the particular values assigned to them. This inevitably requires analytic or semi-analytic understanding of the underlying processes. Anything less is unsatisfactory.

2. Redshift distortions

Viewed in redshift space, which is the only three-dimensional view we have, the universe looks anisotropic: the distribution of galaxies is elongated in what have been called "fingers-of-god" pointing toward us (a phrase probably attributable to Jim Peebles). These fingers-of-god appear strongest where the galaxy density is largest (see Fig. 5), and are attributable to the extra "peculiar" (ie: non-Hubble) component of velocity in the galaxy clusters. This manifests itself as density-correlated radial noise in the radial velocity map.

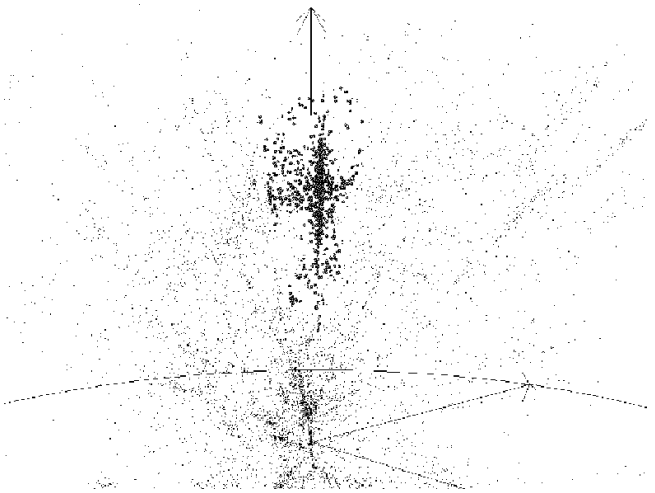

FIG. 5 A view of the three dimensional distribution of galaxies in which the members of the Coma cluster have been highlighted to show the characteristic "finger-of-God" pattern, from Christensen (1996).

Since we know that the real 3-dimensional map should be statistically isotropic, this finger-of-god effect can be filtered out. There are several techniques for doing that: it has become particularly important in the analysis of the vast 2dF (2 degree Field) and SDSS (Sloan Digital Sky Survey) surveys (Tegmark *et al.*, 2002). The earliest discussion of this was probably Davis and Peebles (1983).

There is another important macroscopic effect to deal with resulting from large scale flows induced by the large scale structure so clearly seen in the CfA-II Slice (de Lapparent *et al.*, 1986). Matter is systemically flowing out of voids and into filaments; this superposes a density-dependent pattern on the redshift distribution that is not random noise as in the finger-of-god phenomenon. This distorts the map (Hamilton, 1998; Kaiser, 1987; Sargent and Turner, 1977). As this distortion enhances the visual intensity of galaxy walls, which are perpendicular to the line-of-sight, it is called "the bull's-eye effect" (Praton *et al.*, 1997).

3. Flux-limited surveys and selection functions

Whenever we see a cone diagram of a redshift survey (see Fig. 6), we clearly notice a gradient in the number of galaxies with redshift (or distance). This artefact is consequence of the fact that redshift surveys are flux-limited. Such surveys include all galaxies in a given region of the sky exceeding an apparent magnitude cutoff. The apparent magnitude depends logarithmically on the observed radiation flux. Thus only a small fraction of intrinsically very high luminosity galaxies are bright enough to be detected at large distances.

For the statistical analyses of these surveys there are two possible approaches:

1. *Extracting volume-limited samples.* Given a distance limit, one can calculate, for a particular cosmological model, the minimum luminosity of a galaxy that still can be observed at that distance, considering the flux limit of the sample. Galaxies in the whole volume fainter that this luminosity will be discarded. The remaining galaxies form a homogeneous sample, but the price paid —ignoring much of the hard-earned amount of redshift information— is too high.

2. *Using selection functions.* For some statistical purposes, such as measuring the two-point correlation function, it is possible to use all galaxies from the flux-limited survey provided that we are able to assign a weight to each galaxy inversely proportional to the probability that a galaxy at a given distance r is included in the sample: this is dubbed *the selection function* $\varphi(r)$. This quantity is usually derived from the luminosity function, which is the number density of galaxies within a given range of luminosities. A standard fit to the observed luminosity function is provided by the Schechter function (Schechter, 1976)

$$\phi(L)dL = \phi_* \left(\frac{L}{L_*}\right)^{\alpha} \exp\left(-\frac{L}{L_*}\right) d\left(\frac{L}{L_*}\right), \qquad (5)$$

where ϕ_* is related to the total number of galaxies and the fitting parameters are L_*, a characteristic luminosity, and the scaling exponent α of the power-law dominating the behavior of Eq. 5 at the faint end.

The problem with that approach is that the luminosity function has been found to depend on local galaxy density and morphology. This is a recent discovery and has not been modelled yet.

4. Corrections to redshifts and magnitudes

The redshift distortions described earlier can be accounted for only statistically (Tegmark *et al.*, 2002); there is no way to improve individual redshifts. However, individual measured redshifts are usually corrected for our own motion in the rest frame determined by the cosmic background radiation. This motion consists of several components (the motion of the solar system in the Galaxy, the motion of the Galaxy in the Local Group (of galaxies), and the motion of the Local Group with respect to the CMB rest frame). It is usually lumped together under the label "LG peculiar velocity" and its value is $v^{LG} = 627 \pm 22$ km s^{-1} toward an apex in the constellation of Hydra, with galactic latitude $b = 30° \pm 3°$ and longitude $l = 276° \pm 3°$ (see, e.g., Hamilton (1998)). If not corrected for, this velocity causes a so-called "rffect" (Kaiser, 1987), an apparent dipole density enhancement in redshift space. Application of this correction has several subtleties: see Hamilton (1998).

Most corrections to measured galaxy magnitudes are usually made during construction of a catalog, and are specific to a catalog. There is, however, one universal correction: galaxy magnitudes are obtained by measuring the flux from the galaxy in a finite width bandpass. The spectrum of a far-away galaxy is redshifted, and the flux responsible for its measured magnitude comes from different wavelengths. This correction is called the "K-correction" (Humason *et al.*, 1956);

the main problem in calculating it is insufficient knowledge of spectra of far-away (and younger) galaxies. In addition, directional corrections to magnitudes have to be considered due to the fact that the sky is not equally transparent in all directions. Part of the light coming from extragalactic objects is absorbed by the dust of the Milky Way. Due to the flat shape of our galaxy, the more obscured regions correspond to those of low galactic latitude, the so-called zone of avoidance, although the best way to account for this effect is to use the extinction maps elaborated from the observations (Schlegel *et al.*, 1998).

C. The first generation of redshift surveys

1. CfA surveys

The first CfA redshift survey was undertaken by Huchra *et al.* (1983) who mapped some 2400 galaxies down to $m \simeq 14.5$ taken from the Zwicky catalog. This survey was too sparse to show definite structure.

The first survey to truly reflect the cosmic structure was the first CfA-II slice of de Lapparent *et al.* (1986), the "Slice of the Universe" (the smallest wedge in Fig. 6). The slice showed very clearly the "bubbly" nature of the large-scale structure, as the authors defined it. This important discovery generated a lot of publicity: cartoons appeared in newspapers depicting females with their arms in a sink full of soap bubbles, and the *Encyclopaedia Britannica* was updated to include a picture of the slice.

Prior to that there had been smaller surveys, such as the Perseus-Pisces region survey of Giovanelli and Haynes (1985) and the Coma-A1367 survey of Chincarini *et al.* (1983). These surveys had revealed rich structures in the distribution of galaxies, similar to Zel'dovich's predicted pancakes and voids. But since they were restricted to a volume around a major cluster of galaxies they could not be thought of as being representative of the universe as a whole.

At first glance it may seem that similar critique applies also to the CfA surveys, since the first CfA slice (de Lapparent *et al.*, 1986) was indeed centered on the Coma cluster. However, the breadth of the slice (some 120 degrees on the sky) samples a far greater volume, and it was very deep for that time, extending to about $150h^{-1}$Mpc. The slice also contains an unusual number of rich galaxy clusters. Subsequent surveys, the following CfA slices and the ESO Southern survey (da Costa *et al.*, 1991) amply confirmed the impression given by the CfA slice.

The main source for redshifts during those years was 'Zcat', a heterogeneous compilation of galaxy redshifts by J. Huchra. But it took many years before the data from the CfA slices entered the public domain. This was unfortunate since many other groups would have liked to try their own analysis techniques on such a well defined sample. By the time that the data became available there existed already more substantial surveys with publicly available data and much of the impetus of the CfA slices, apart from the fine work done by the Harvard group itself, was lost.

The work to improve and extend the CfA surveys has continued. The Century Survey (Geller *et al.*, 1997) covers the central 1° region of the famous CfA-II slice, but is much deeper, extending to $R = 16.1$ in the apparent magnitude and to $450h^{-1}$Mpc in space. The final CfA catalog is the Updated Zwicky Catalog (Falco *et al.*, 1999) that includes uniform measurements of almost all (about 19,000) galaxies of the Zwicky catalog (with the magnitude limit of $m_{Zw} \approx 15.5$) in the northern sky. Nowadays catalogs are made public as soon as possible; the CfA redshift catalogs can be obtained from the web-page of the Smithsonian Astronomical Observatory Telescope Data Center (http://tdc-www.harvard.edu/).

2. SSRS and ORS

The Southern Sky Redshift Survey (da Costa *et al.*, 1991) was meant to complement the original CfA survey, mapping galaxies in the southern sky. It includes almost 2000 redshifts; the followup survey, the extended SSRS (da Costa *et al.*, 1998) with about 5400 redshifts mirrored the Second CfA survey for the southern sky. These catalogs were mostly used for comparison with the CfA survey results; they were made public at once and produced many useful results. Presently they are available from the Vizier database (http://vizier.u-strasbg.fr).

The Optical Redshift Survey (Santiago *et al.*, 1995), had a depth of $80h^{-1}$Mpc, similar to the first CfA survey, but attempted a complete coverage of the sky (except for the dusty avoidance zone around the galactic equator). They measured about 1300 new redshifts, including about 8500 redshifts in total. This survey was heavily exploited to describe the nearby density fields, to estimate the luminosity functions, galaxy correlations, velocity dispersions etc. The catalog and the publications can be found in http://www.astro.princeton.edu/∼strauss/ors/.

3. Stromlo-APM and Durham/UKST redshift surveys

The Stromlo-APM redshift survey (Loveday *et al.*, 1996) is a sparse survey (1 in 20) of some 1800 optically selected galaxies brighter than the apparent magnitude limit $B \approx 17$ taken from the APM survey of the Southern sky. As the APM survey (Maddox *et al.*, 1990) itself, the Stromlo-APM survey was an important data source and generated several important results on correlation functions in real and redshift space, power spectra, redshift distortions, cosmological parameters, bias and so on. It was eventually put into the public domain, although rather too late to be of much use to any third party investigators.

The APM survey was also used to generate a galaxy cluster catalog. The APM cluster redshift catalog (Dalton *et al.*, 1997) was the first objectively defined cluster catalog. It not only provided important data on the distribution of clusters, it also provided an assessment of the reliability of the only cluster source available before that, the Abell cluster catalog.

The Durham/UKST redshift survey (Ratcliffe *et al.*, 1998) measured redshifts for about 2500 galaxies around the South

Galactic Pole. The depth of the survey was similar to that of the Stromlo-APM survey, and it was also a diluted survey sampling 1 galaxy in 3.

These catalogs can be found now at the Vizier site (see above).

4. IRAS redshift samples: PSCz

The story of the IRAS (Infrared Astronomical Satellite) redshift catalogs stresses the importance of having a good base photometric catalog before starting to measure redshifts. As galactic absorption in infrared is much smaller than in the optical bands, the IRAS Point Source Catalog (PSC) covers uniformly almost all of the sky. This catalog was used to select galaxies for redshift programs, which extended down to successively smaller flux limits: the 2 Jy survey of Strauss *et al.* (1992) with 2658 galaxies; the 1.2 Jy survey of Fisher *et al.* (1995) added 2663 galaxies; and the 0.6 Jy sparse-sampled (1 in 6) QDOT survey of Lawrence *et al.* (1999) with 2387 galaxies. This culminated in the PSCz survey of some 15000 galaxies by Saunders *et al.* (2000), which includes practically all IRAS galaxies within the 0.6 Jy flux limit.

The IRAS redshift catalogs have been used for the usual battery of large-scale studies, but their main advantage is their full-sky coverage (about 84%). This allows using the Wiener-type reconstruction methods to derive the true density and velocity fields, and to get an independent estimate of the biasing parameter. The first fields to be studied were taken from the 2 Jy survey by Yahil *et al.* (1991), the last fields came from the PSCz survey by Branchini *et al.* (1999) and Schmoldt *et al.* (1999).

The PSCz survey has also been used for fractal studies. Although the IRAS samples are not too deep (PSCz extends to about $200h^{-1}$Mpc), Pan and Coles (2000) found that multifractal analysis shows a definite crossover to homogeneity already before this scale.

5. ESO Deep Slice and the Las Campanas redshift survey

The ESO Deep Slice (Vettolani *et al.*, 1998) measured redshifts of 3300 galaxies down to the blue magnitude $b_J = 19.4$ in the B_J, R, I photometric system (Gullixson *et al.*, 1995). The surveyed region is a $1^\circ \times 22^\circ$ strip of depth about $600h^{-1}$Mpc. The most interesting discussion that this data caused was about the fractal nature of the large-scale galaxy distributions. While Scaramella *et al.* (1998) found the correlation dimension $D = 3$, Joyce *et al.* (1999) showed that a more reasonable choice of the K-correction (redshift-dependent apparent dimming of galaxies) gave a clearly fractal $D = 2$ correlation dimension.

The Las Campanas Redshift Survey (Shectman *et al.*, 1996) had a similar geometry, six thin parallel slices ($1.5^\circ \times 90^\circ$) with the depth about $750h^{-1}$Mpc ($z \approx 0.25$). The survey team measured redshifts of about 24000 galaxies in these slices. This was the first deep survey of sufficient volume that it could be used to test if our knowledge of the nearby Universe was sufficient to describe more distant regions. The usual tests included the luminosity functions (these were found to depend on galaxy density and morphology), second- and third-order correlation functions, power spectra, and fractal properties. A catalog of groups of galaxies was generated. The survey results were quickly made public: the general interest in the data was high and close to a hundred papers have been published using these data.

D. Recent and on-going Surveys

1. 2dF galaxy redshift survey

The 2dF multi-fiber spectrograph on the 3.9m Anglo-Australian Telescope is capable of observing up to 400 objects simultaneously over a field of view some 2 degrees in diameter, hence the name of the survey. The sample of galaxies targeted for having their redshifts measured consists of some 250,000 galaxies located in extended regions around the north and south Galactic poles. The source catalog is a revised APM survey. The galaxies in the survey go down to the magnitude $b_J = 19.45$. The median redshift of the sample is $z = 0.11$ and redshifts extend to about $z \simeq 0.3$. In mid-2001 the survey team released the data on the first 100,000 galaxies, and published also an interim report on the analysis of some 140,000 galaxies: Peacock *et al.* (2001) and Percival *et al.* (2001).

The survey is already complete, and the resulting correlation functions, redshift distortions and pairwise velocity dispersions (Hawkins *et al.*, 2003) demonstrate the quality of the data set. The 2dFGRS currently provides us with the best estimates for a large number of cosmological parameters describing the population of galaxies. Not only can we determine clustering properties of the sample as a whole, but the sample can be broken down by galaxy absolute brightness or by morphological type (Percival *et al.*, 2004). The surveys's web page is http://www.mso.anu.edu.au/2dFGRS/.

2. Sloan digital sky survey

Hot on the heels of the 2dF survey is an even larger survey: the Sloan Digital Sky Survey (SDSS). The survey team has close to two hundred members from 13 institutions in U.S., Europe, and Japan, and uses a dedicated 2.5 m telescope. The initial photometric program is measuring the positions and luminosities of about 10^8 objects in π sterradians of the Northern sky, and the follow-up spectroscopy is planned to give redshifts of about 10^6 galaxies and 10^5 quasars. Good descriptions of the survey can be found in Loveday (2002) and on the surveys's web page (http://www.sdss.org/).

The first official data release was done in 2003, but the astronomical community had already have the chance to see and use the data from a preliminary Early Data Release (Stoughton *et al.*, 2002). These data and the data from the commissioning phase of the project have served as a basis for more than one hundred papers on such diverse subjects as the study of asteroids, brown dwarf stars in the vicinity of the Sun, remnants of destroyed satellites of our Galaxy, star formation

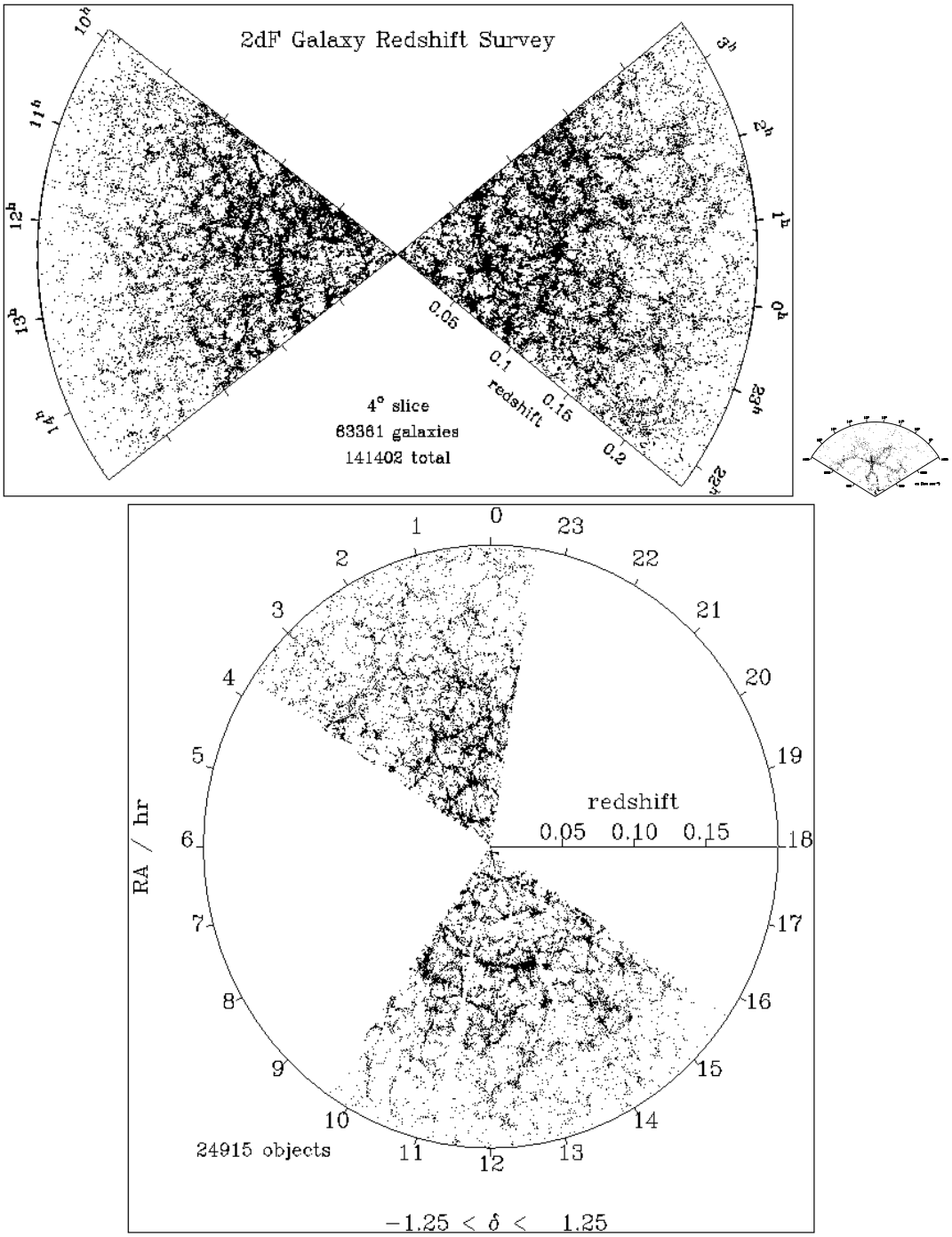

FIG. 6 The top diagram shows two slices of 4° width and depth $z = 0.25$ from the 2dF galaxy redshift survey, from Peacock *et al.* (2001). The circular diagram at the bottom has a radius corresponding to redshift $z = 0.2$ and shows 24,915 galaxies from the SDSS survey, from (Loveday, 2002)). As an inset on the right, the first CfA-II slice from de Lapparent *et al.* (1986) is shown to scale.

rates in galaxies, galaxy luminosity functions, and, of course, on the statistics of the galaxy distribution.

The main difference between the 2dF and the SDSS surveys, apart of their data volume and sky coverage, is the fact that they are based on different selection rules. While the 2dF survey is a blue-magnitude limited survey with $b_{\rm lim} = 19.45$, the limiting magnitude of the SDSS survey is red $r_{\rm lim} = 17.77$. This causes considerable differences in galaxy morphologies of the two surveys. Also, while the depths of the main surveys are similar ($z \approx 0.25$), a part of the SDSS survey, including about 10^5 luminous red galaxies, will reach redshifts $z \approx 0.5$.

3. 2MASS and 6dF

The Two Micron All Sky Survey (2MASS) has scanned the whole sky in three different near-infrared bands. The Extended Source Catalog (XSC) is the 2MASS galaxy catalog (Jarrett, 2004) and contains more then 1.5 million galaxies, mapping rather well the zone of avoidance, The view of our local universe provided by 2MASS is shown in Fig. 7.

The 6dF galaxy survey (Jones *et al.*, 2004) targeted on the 2MASS galaxy catalog (XSC) will encompass twice the volume of the PSCz and will contain ten times more galaxies, allowing combined knowledge of galaxy masses and redshift. It will be the best sample for studies of the peculiar velocity field, allowing a better understanding of the relation of galaxy clustering with mass, and hence providing important clues to understand how bias depends on the scale.

4. Deep spectroscopic and photometric surveys

Deep spectroscopic surveys such as the Canadian Network for Observational Cosmology (CNOC2) (Yee *et al.*, 2000), DEEP2 (Davis *et al.*, 2003), and the Visible Imaging Multi-Object Spectrograph (VIRMOS-VLT) survey (Le Fèvre *et al.*, 2003) have allowed the study of the evolution of clustering with redshift and with various morphological properties of galaxies (Carlberg *et al.*, 2000; Coil and DEEP2 Team, 2003). Nevertheless, it is extremely difficult to measure redshifts of very faint objects. The present limit reached making use of the largest ground-based telescopes is about $I \simeq 24$. An alternative to spectroscopy, is the poor man z- machine (Koo, 1985), provided by multi-wavelength imaging.

Following the pioneering work of Baum (1962) and Koo (1985), Fernández-Soto *et al.* (1999) have shown that it is possible to reliably estimate redshifts using CCD images at different wavebands —the so called photometric redshifts—. This technique is particularly useful when mapping the very distant universe because galaxies in deep surveys could not be spectroscopically observable. Bayesian techniques have been introduced to improve the accuracy of the photometric redshift estimation (Benítez, 2000).

Different surveys reaching extremely large depths are providing us with the possibility of analyzing the evolution of clustering with cosmic time. We can mention the COMBO17 survey (Classifying Objects by Medium-Band Observations) which lists photometry in 17 passbands (Wolf *et al.*, 2004), the Calar Alto Deep Imaging Survey (CADIS), used by Phleps and Meisenheimer (2003) to show how the clustering strength grows from $z = 1$ to the present epoch and its dependence on morphological type, and the recently released Great Observatories Origins Deep Survey (GOODS) described in Giavalisco *et al.* (2004). The SDSS provides also photometric information in five bands allowing the measurement of photometric redshifts for a volume-limited sample containing more than 2 million galaxies within the range $0.1 < z < 0.3$. Analyzing the angular two-point correlation function of this survey, Budavári *et al.* (2003) have found an interesting bimodal behavior between *red* elliptical-like galaxies and *blue* galaxies.

The recent project named the ALHAMBRA-survey (Advanced Large, Homogeneous Area Medium Band Redshift Astronomical survey) is being carried out by Moles and collaborators using the 3.5m Calar Alto telescope. The photometric survey will cover an area of eight square degrees. Imaging will be performed using 20 optical filters plus three standard bands in the near infrared. It is expected to collect about 600,000 photometric galaxy redshifts with an accuracy of $\Delta z < 0.015(1 + z)$. This photometric survey, midway between the wide-angle spectroscopic surveys and the narrow imaging surveys, is deep enough and wide enough to be extremely useful for all kind of studies involving cosmic evolution.

E. The radio, X-ray and γ-ray skies

The 1950's was a great era for cataloguing radio sources, much of the work being done at Cambridge in England (with the 2C, 3C, etc. surveys) and at Parkes in Australia. The surveys were done at considerably different frequencies and gave disparate views of the source counts. This had a strong influence on the Steady State versus Big Bang debate, each survey being used to support a different cosmological hypothesis.

The sources in early surveys were randomly distributed over the sky (for instance, Holden, 1966 on the Third Cambridge Catalog and Payne, 1967 on the southern counterpart). This remained true for later surveys at low frequencies, which found, for the most part, intrinsically very bright sources at somewhat larger distances (for instance, Webster, 1976 analyzing the Fourth Cambridge and Greenbank surveys, and Masson, 1979 on the Sixth Cambridge Catalog). Indeed it remains true down to the present day (Trimble and Aschwanden, 2001), for the low-frequency surveys that pick out large, bright, steep-spectrum, extended double sources: Artyukh (2000); Venturi *et al.* (2000) reported that they did not even identify the Shapley concentration). What this means is that, on average, there is only one of these sources in each of the largest-scale structures to be found in the local universe. The absence of clustering is, therefore, in some sense evidence for the existence of "largest structures," though Artyukh and Venturi *et al.* note that mergers of small groups into large clusters and superclusters may well turn off fainter radio sources that would otherwise reveal intermediate structure.

In contrast, higher frequency surveys that yield intrinsically fainter radio galaxies find that they are clustered very much like radio-quiet galaxies of the same Hubble types (Cress *et al.*, 1996 on the Faint Images of the Radio Sky at Twenty-cm (FIRST) survey from the Very Large Array (VLA), and Magliocchetti *et al.*, 1998 a further analysis of FIRST, showing that the distribution of those radio sources in space is consistent with their having grown by gravitational instabilities from Gaussian initial conditions). Returning to the Shapley concentration, Venturi *et al.* (2002) found no fewer than 124 radio sources there.

Distant radio sources (of which quasars are an important

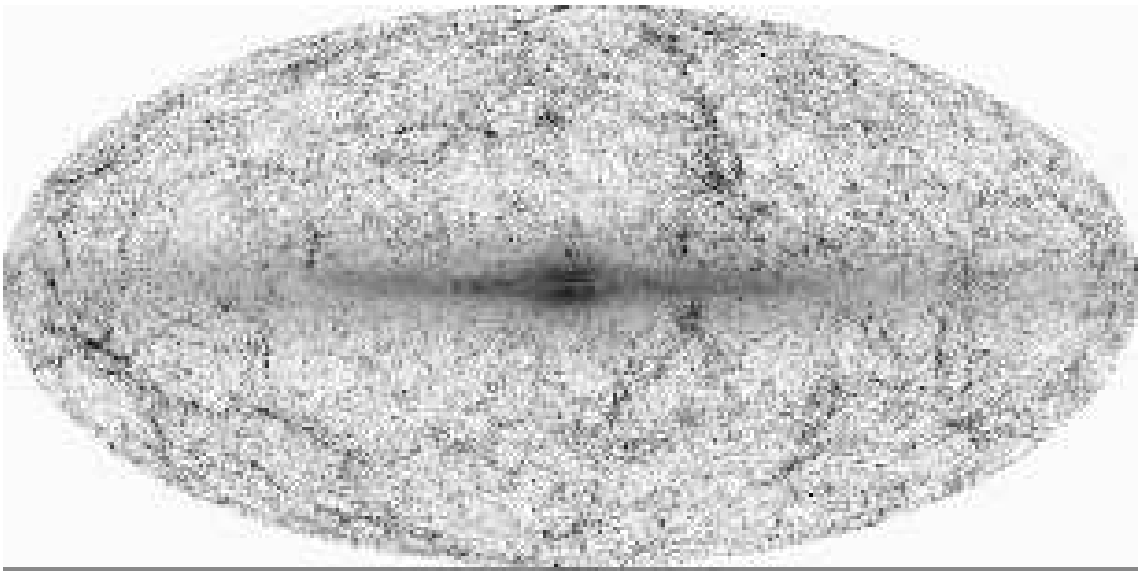

FIG. 7 The near-infrared view of the local universe provided by the 2MASS survey. Beyond the Milky Way lying at the Galactic equator, more than 1.5 million galaxies are depicted using a grey-scale code based on their photometrically deduced redshift, from Jarrett (2004).

sort) are rather sparsely distributed throughout the Universe and are consequently not good indicators of large scale structure. It is therefore not surprising that radio source catalogs provide little evidence for the large scale clustering.

Galaxy clusters are prominent features of the X-ray sky that can provide a good measure of the large scale clustering. X-ray selected samples of clusters are less prone to bias than catalogs for clusters selected from maps of the galaxy distribution. One problem, however, is that the selection criteria for galaxy clusters selected from X-ray surveys (Borgani and Guzzo, 2001) are quite different from the selection criteria for clusters selected from optically scanned photographic plates (Dalton *et al.*, 1997) and it is not so easy to relate studies based on the two sources of data.

The REFLEX (ROSAT-ESO Flux Limited X-ray) cluster survey contains 449 clusters, covering an area of 4.24 steradians in the southern hemisphere ($\delta < 2.5^\circ$). It is complete at $\geq 90\%$, down to a nominal flux limit of 3×10^{-12} erg s^{-1} cm^{-2} in the $0.1 - 2.4$ keV band. REFLEX, as other cluster samples, shows unambiguously very large-scale inhomogeneities that appear when the clustering power is measured and compared with that of galaxies at the same scales (Guzzo, 2002).

F. Distribution of quasars and Ly-α clouds

The spectra of quasars are populated by narrow absorption lines from intervening gas clouds along the line of sight (the Ly-α forest). Owing to the great redshift of most quasars these absorption clouds provide an important probe of clustering at large distances and at times long in our past.

Wu *et al.* (1999) used the large-scale uniformity of the Ly-α forest to argue against fractal distribution of matter. Recently, Croft *et al.* (2002) showed that it is possible to estimate the full 3-D power spectrum of density fluctuations $P(k)$ from the (one-dimensional) Ly-α flux power spectrum. This is extremely important, as it allows us to check for theoretical predictions at large redshifts ($z \approx$2–4). It also allows us to recover the linear (post-recombination) power spectrum for small scales, which have turned nonlinear by now.

Lines of sight to quasar pairs, be they optical pairs or pairs that are a consequence of gravitational lensing, provide additional clues to the clustering transverse to the line of sight (Wu *et al.*, 1999).

The statistical analysis of the distribution of quasars and Ly-α clouds has provided additional evidence for the large scale homogeneity in the universe (Andreani *et al.*, 1991; Carbone and Savaglio, 1996).

G. The cosmic microwave background

The importance of the CMB anisotropy measurements cannot be over-emphasized and would warrant an entire review by itself. From the point of view of this article we are concerned with knowing the initial conditions for galaxy formation and the parameters of the cosmological framework within which galaxy formation takes place. Given that data, the task is to derive the currently observed clustering properties of galaxies in the Universe.

1. Structure before our eyes

Arguably the most important observation in the study of clustering is the recent measurement of the structure in the cosmic microwave background radiation at the time of recombination. This structure was predicted independently by Silk (1967) and by Sachs and Wolfe

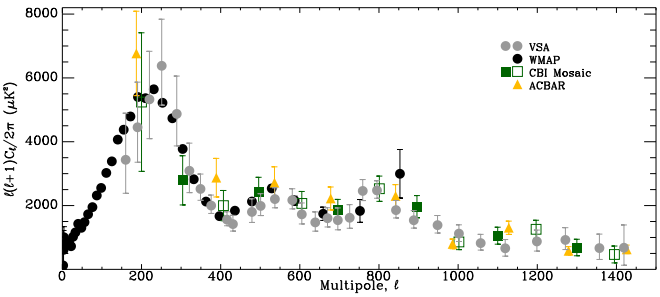

FIG. 8 The agreement between the estimated power spectrum of the CMB anisotropies from four different experiments with similar sensitivity, from Dickinson *et al.* (2004).

(1967), although the phenomenon is generally referred to as the "Sachs-Wolfe" effect. Understanding the details of how the structure in the microwave background arises in any of a vast number number of cosmological models has been a cosmic folk-industry spanning some 30 years. The results are encapsulated in a run-it-yourself computer program of Zaldarriaga and Seljak, 2000 (see http://physics.nyu.edu/matiasz/CMBFAST/cmbfast.html).

The structure was first seen at about 7° in angular resolution in the data of the COBE satellite DMR experiment (Bennett et al, 1996). Smaller structure has been detected in recent high angular resolution experiments with names like DASI (Leitch *et al.*, 2002; Pryke *et al.*, 2002), MAXIMA-1 (Balbi *et al.*, 2000; Hanany *et al.*, 2000; Lee *et al.*, 2001) and BOOMERANG-98 (de Bernardis *et al.*, 2000; Lange *et al.*, 2001; Netterfield *et al.*, 2002), and in the WMAP first-year full-sky data (Bennett *et al.*, 2003). An analysis of the cosmological conclusions to be drawn from the combination of these is given by Jaffe *et al.* (2001) and by Spergel *et al.* (2003); an example of present data sets and the curves fitted to them is shown in Fig. 8 where, in addition to the WMAP power-spectrum, several other recent experiments are shown (VSA analyzed by Dickinson *et al.* (2004), CBI (Mason *et al.*, 2003) and ACBAR (Kuo *et al.*, 2004)), having similar sensitivity, but being different in the frequency range and observing techniques.

Here we observe unambiguously the structure in the gravitational potential that will lead to the birth and clustering of galaxies and clusters of galaxies as we see them today. We also observe structure on scales far larger than can be traced by galaxies.

The units in Fig. 8 could use a little bit of explanation. As the sky we see can be thought of as a surface of a sphere, the distribution of temperature on the sky is analysed into scales using Legendre polynomials $Y_l^m(\theta,\phi)$. A polynomial of order l picks out structure on an angular scale that is roughly, in degrees,

$$\theta^\circ \approx \frac{180^\circ}{l} \tag{6}$$

This corresponds to structure on a linear scale today of

$$L = \frac{2\pi c}{H_0 \Omega_m^{0.4} \ell} \approx \frac{19000}{\ell} \Omega_m^{-0.4} h^{-1} \mathrm{Mpc}. \tag{7}$$

for a flat universe with $\Omega_m + \Omega_\Lambda = 1$ (Vittorio and Silk, 1992). The range of l-values covered by current experiments range over about two decades:

$$10 < l < 1500 \tag{8}$$

with the limit of higher l-values being pushed upward all the time. The low resolution end is from the COBE and WMAP data (Bennett *et al.*, 1996; Bennett *et al.*, 2003) and reveals inhomogeneities on scales in excess of $100h^{-1}$ Mpc.

Notice that the highest resolution data still only cover linear scales in excess of around $30h^{-1}$ Mpc and so we do not yet see the initial condition for the scales over which the two-point galaxy clustering correlation function is significantly greater than zero. We are just seeing the scales where rich cluster clustering may be significant. The prominent peak in the spectrum at $l \sim 250$ corresponding to scales of around $50h^{-1}$ Mpc is intriguing. We must not forget, however, that this is a peak in a normalized spectrum; in the real matter $P(k)$ these peaks are much less pronounced. There is evidence of oscillations in the observed power spectra of clusters and galaxies, but current surveys are not able yet to detect such structure with confidence (Elgarøy *et al.*, 2002; Miller *et al.*, 2002a).

2. Defining the standard model

The presence of significant peaks in the angular distribution of the cosmic microwave background strongly constrains the global parameters that describe our Universe. If these data are combined with data from other sources, such as local determinations of the Hubble constant and observations of very distant supernovae (Perlmutter *et al.*, 1999; Riess *et al.*, 1998), we arrive at the so-called *concordance model* (Tegmark *et al.*, 2001). We hasten to add that this is not a term we invented: it might have been OK to use the term *standard model*, but the high energy physicists got there first. The actual values of the parameters in the concordance model depends on whose paper we read: there is a little disaccord here, though it would seem to be relatively minor. It all depends on what prior knowledge is assumed when making fitting the model to the data. The error bars are impressively small.

3. Initial conditions for galaxy formation

One of the best determined parameters is the slope n of the power spectrum of the pre-recombination inhomogeneities. It was suggested by Harrison and by Zel'dovich that $n = 1$ on the grounds that (a) the spectrum had to be a power law (what else could it be?) and that (b) this value of the slope was the value that did the minimal violence to the geometry of space-time on either the large or small scales. Following on Guth's brilliant notion of inflationary cosmology (Guth, 1981), many subsequent revisions of the inflationary model and theories for the origin of cosmic fluctuations gave physical reasons why we should have $n = 1$ (e.g.: Guth and Pi (1982), Starobinskii (1982), Linde (1994, 1982, 1983)).

The DASI experiment (Pryke *et al.*, 2002) gives

$$n = 1.01^{+0.08}_{-0.06} \tag{9}$$

where the error bars are 68% confidence limits. This result comes from fitting the DASI data alone, making typical prior assumptions about such things as the Hubble constant. The recent WMAP data gives a value

$$n = 0.99 \pm 0.04 \tag{10}$$

(Spergel *et al.*, 2003). (This latter value comes from the WMAP data alone, no other data is taken into account.) Other similar numbers come from Wang *et al.* (2002) and Miller *et al.* (2002b).

It is perhaps appropriate to point out that this fit comes from data on scales bigger than the scale of significant galaxy clustering and that it is a matter of belief that the primordial power law continued in the same manner to smaller scales. In fact, more complex inflationary models predict a slowly varying exponent (spectral index) (see, e.g., Kosowsky and Turner (1995)); this is in accordance with the WMAP data. The scales which are relevant to the clustering of galaxies are just those scales where the effects of the recombination process on the fluctuation spectrum are the greatest. We believe we understand that process fully (Hu *et al.*, 2001, 1997) and so we have no hesitation in saying what are the consequences of having an initial $n = 1$ power spectrum. That, and the success of the N-body experiments, provide a good basis for the belief that $n \approx 1$ on galaxy clustering scales. Anyway, it is probable that the Sunyaev-Zel'dovich effect (Sunyaev and Zel'dovich, 1980) will dominate on the scales we are interested in so we may never see the recombination-damped primordial fluctuations on such scales.

We therefore have a classical initial value problem: the difficulty lies mainly in knowing what physics, subsequent to recombination, our solution will need as input and knowing how to compare the results of the consequent numerical simulations with observation. CMB measurements can also give us valuable clues for these later epochs in the evolution of the universe. A good example is the discovery of significant large-scale CMB polarization by the WMAP team (Kogut *et al.*, 2003) that pushes the secondary re-ionization (formation of the first generation of stars) back to redshifts $z \approx 20$.

V. MEASUREMENTS OF CLUSTERING

A. The discovery of power-law clustering

The pioneering work of Rubin and Limber has already been mentioned. These early authors were limited by the nature of the catalogs that existed at the time and the means to analyze the data – there were no computers!

It was Totsuji and Kihara (1969) and, independently, Peebles (1974b) who were first to present a computer-based analysis of a complete catalog of galaxies. Totsuji and Kihara used the published Lick counts in cells from Shane and Wirtanen (1967), while Peebles and coworkers analysed a number of catalogs: the Reference catalog of Bright Galaxies, the Zwicky catalog, the Lick catalog and later on the very deep Jagellonian field (Peebles, 1975; Peebles and Groth, 1975; Peebles and Hauser, 1974). All this work was done on the projected distribution of galaxies since little or no redshift information was available.

The central discovery was that the two-point correlation function describing the deviation of the galaxy distribution from homogeneity scales like a simple power law over a substantial range of distances. This result has stood firm through numerous analyses of diverse catalogs over the subsequent decades.

The amplitudes of the correlation functions calculated from the different catalogs were found to scale in accordance with the nominal depth of the catalog. This was one of the first direct proofs that the Universe is homogeneous. Before that we knew about the isotropy of the galaxy distribution at different depths and could only infer homogeneity by arguing that we were not at the center of the Universe.

B. The correlation function: galaxies

1. Definitions and scaling

The definition of the correlation function used in cosmology differs slightly from the definition used in other fields. In cosmology we have a nonzero mean field (the mean density of the Universe) superposed on which are the fluctuations that correspond to the galaxies and galaxy clusters. Since the Universe is homogeneous on the largest scales, the correlations tend to zero on these scales.

On occasion, people have tried to use the standard definition and in doing so have come up with anomalous conclusions.

The right definition is: In cosmology, the 2-point galaxy correlation function is defined as a measure of the excess probability, relative to a Poisson distribution, of finding two galaxies at the volume elements dV_1 and dV_2 separated by a vector distance $\mathbf{r}$:

$$dP_{12} = n^2[1 + \xi(\mathbf{r})]dV_1 dV_2, \tag{11}$$

where n is the mean number density over the whole sample volume. When homogeneity[5] and isotropy are assumed $\xi(\mathbf{r})$ depends only on the distance $r = |\mathbf{r}|$. From Eq. (11), it is straightforward to derive the expression for the conditional probability that a galaxy lies at dV at distance r given that there is a galaxy at the origin of $\mathbf{r}$.

$$dP = n[1 + \xi(r)]dV. \tag{12}$$

Therefore, $\xi(r)$ measures the clustering in excess ($\xi(r) > 0$) or in defect ($\xi(r) < 0$) compared with a random Poisson point distribution, for which $\xi(r) = 0$. It is worth to mention that in statistical mechanics the correlation function normally used is $g(r) = 1+\xi(r)$ which is called the *radial distribution function*

[5] This property is called stationarity in point field statistics.

(McQuarrie, 1999). Statisticians call this quantity the *pair correlation function* (Stoyan and Stoyan, 1994). The number of galaxies, on average, lying at a distance between r and $r + dr$ from a given one is $ng(r)4\pi r^2$.

A similar quantity can be defined for projected catalogs: surveys compiling the angular positions of the galaxies on the celestial sphere. The angular two-point correlation function, $w(\theta)$, can be defined by means of the conditional probability of finding a galaxy within the solid angle $d\Omega$ lying at an angular distance θ from a given galaxy (arbitrarily chosen):

$$dP = \mathcal{N}[1 + w(\theta)]d\Omega, \tag{13}$$

Now, $\mathcal{N}$ is the mean number density of galaxies per unit area in the projected catalog. Since the first available catalogs were two-dimensional, with no redshift information, $w(\theta)$ was measured before any direct measurement of $\xi(r)$ was possible. Nevertheless, $\xi(r)$ can be inferred from its angular counterpart $w(\theta)$ by means of the Limber equation (Limber, 1954; Rubin, 1954) which provides an integral relation between the angular and the spatial correlation function for small angles,

$$w(\theta) = \int_0^\infty y^4\phi^2(y)\,dy \int_0^\infty \xi\left(\sqrt{x^2 + y^2\theta^2}\right)\,dx. \tag{14}$$

Here y is the comoving distance and $\phi(y)$ is the radial selection function normalized such that $\int \phi(y)y^2dy = 1$. If $\xi(r)$ follows a power law $\xi(r) = (r/r_0)^{-\gamma}$, it is straightforward to see that the angular correlation function is also a power law, $w(\theta) = A\theta^{1-\gamma}$ (Peebles, 1980). Totsuji and Kihara (1969) were the first to derive a power-law model for $\xi(r)$ on the basis of the angular data. Their canonical value for the scaling exponent $\gamma = 1.8$ has remained unaltered for more than 30 years. Eq. 14 provides the basis for an important scaling relation. Peebles (1980) has shown that, in a homogeneous universe, $w(\theta)$ must scale with the sample depth D_* as

$$w(\theta) = \frac{1}{D_*}W(\theta D_*) \tag{15}$$

where the function W is an intrinsic angular correlation function which does not depend on the apparent limmiting magnitude of the sample. The characteristic depth D_* is the distance at which a galaxy with intrinsic luminosity L_* is seen at the limiting flux density f, which is in the Euclidean geometry (neglecting expansion and curvature),

$$D_* = \sqrt{\frac{L_*}{4\pi f}}, \tag{16}$$

or, in terms of magnitudes,

$$D_* = 10^{0.2(m_0 - M_*) - 5}h^{-1}\mathrm{Mpc}, \tag{17}$$

where m_0 is the apparent limiting magnitude of the sample. The scaling relation in Eq. (15) can be deduced from the Limber equation (14) assuming that distribution of galaxies is homogeneous on average and therefore $\mathcal{N} \propto D_*^3$. Peebles (1993) has shown that the analysis of the deep catalogs of galaxies on the basis of the scaling law (Eq. 15) argues strongly against an unbounded self-similar fractal distribution of galaxies. In the 1970's and early 1980's a number of catalogs going to a variety of magnitude limits were available and analysed by Peebles and his collaborators. Because of the way the galaxy luminosity function works, most of the galaxies in a catalog fall within a relatively narrow range of distance that depends on the limiting magnitude of the catalog: catalogs reaching to fainter magnitudes are probing the Universe at greater distances.

As the distance increases, the angular scale subtended by a given physical distance decreases. Hence, if the Universe is homogeneous, the two-point angular correlation function of one catalog should look like a rescaled version of the two point angular correlation function of a deeper catalog (see Eq. 15), i.e., for catalogs with varying characteristic distance $w \propto D_*^{-1}$ at a given angular separation θD_*; or, in other words, if we calculate the angular correlation function on two samples, with characteristic depths D_* and D'_*, Eq. 15 implies that $w'((D_*/D'_*)\theta) = (D_*/D'_*)w(\theta)$. The scaling relationship can be predicted precisely, though for catalogs that probe to very great depths it is necessary to be careful of K-corrections and geometric effects due to the cosmological model (Colombo and Bonometto, 2001).

The earliest catalogs available were the de Vaucouleurs catalog of Bright Galaxies, the Zwicky catalog, the Shane–Wirtanen catalog and the Jagellonian Field. Matching their correlation functions provided the first direct evidence for large scale cosmic homogeneity (Groth and Peebles, 1977, 1986). The scaling relation has been confirmed with more recent catalogs, in particular, the APM galaxy survey has povided one of the strongest observational evidences supporting this law (Baugh, 1996; Maddox *et al.*, 1996; Maddox *et al.*, 1990b).

Now we can do much better since we have bigger and better catalogs with partial or complete redshift information. Such catalogs can be divided into magnitude slices and the same test performed on the two point angular correlation function of the slices. The result (Connolly *et al.*, 2002) reproduced in Fig. 9 is as good a vindication of the homogeneity of the Universe as one could wish for. More data will be forthcoming from the 2dF and SDSS surveys.

The scaling properties of the correlation function are usually shown in the form of the correlation integral. For a point distribution the integral expresses the number of neighbors, on average, that an object has within a sphere of radius r. It is given by

$$N(< r) = n\int_0^r 4\pi s^2(1 + \xi(s))ds \tag{18}$$

The distribution is said to follow fractal scaling if within a large range of scales the behavior of $N(< r)$ can be well fitted to a power law

$$N(< r) \propto r^{D_2}, \tag{19}$$

or, alternatively

$$1 + \xi(r) \propto r^{D_2 - 3}. \tag{20}$$

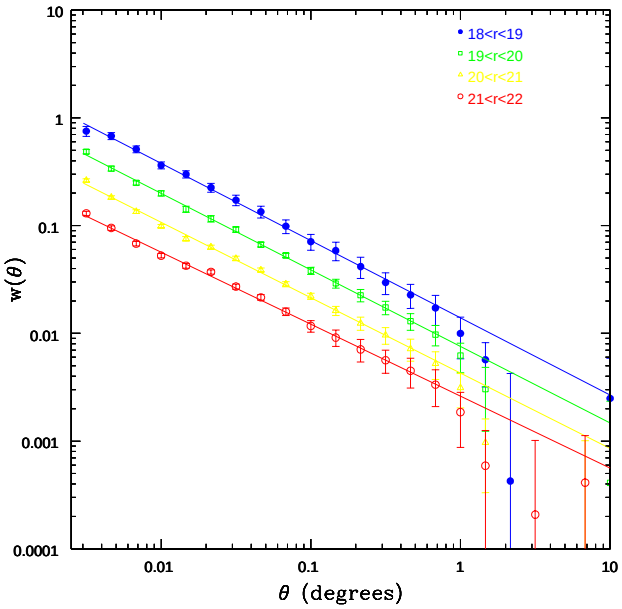

FIG. 9 The angular correlation function from the SDSS as a function of magnitude from Connolly *et al.* (2002). The correlation function is determined for the magnitude intervals $18 < r^* < 19$, $19 < r^* < 20$, $20 < r^* < 21$ and $21 < r^* < 22$. The fits to these data, over angular scales of 1' to 30', are shown by the solid lines.

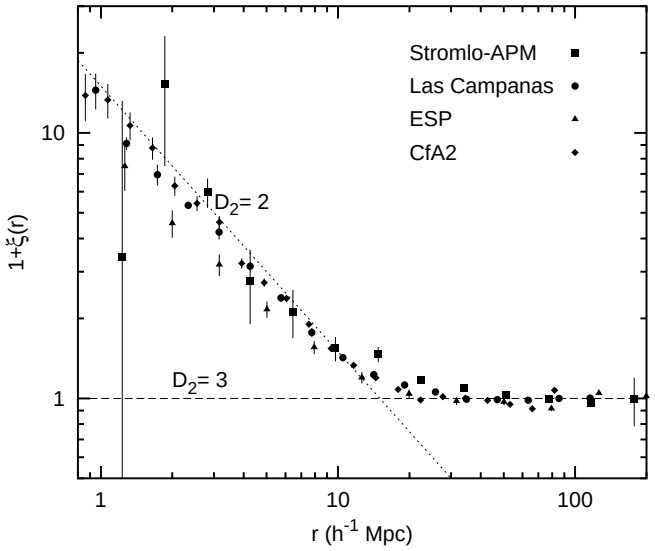

FIG. 10 The correlation function $1 + \xi(r)$ for different samples calculated with different estimators. We can see that the small scale fractal regime is followed by a gradual transition to homogeneity.

where D_2 is the so-called correlation dimension. The scaling range has to be long enough to talk about fractal behavior. However, the term has been used very often for describing scaling behaviors within rather limited scale ranges (Avnir *et al.*, 1998). In Sect. V.B.4 we show recent determinations of D_2 for several galaxy samples at different scale ranges.

2. Estimators

The two-point correlation function $\xi(r)$ can be estimated in several ways from a given galaxy sample. For a discussion of them see, for example, Kerscher *et al.* (2000); Martínez and Saar (2002); Pons-Bordería *et al.* (1999). At small distances, nearly all the estimators provide very similar performance, however at large distances, their performance is not equivalent any more and some of them could be biased. Considering the galaxy distribution as a point process, the two-point correlation function at a given distance r is estimated by counting and averaging the number of neighbors each galaxy has at a given scale. It is clear that the boundaries of the sample have to be considered, because as no galaxies are observed beyond the boundaries, the number of neighbors is systematically underestimated at larger distances. If we do not make any assumption regarding the kind of point process that we are dealing with, the only solution is to use the so-called minus–estimators, the kind of estimators favored by Piertonero and co-workers (Sylos Labini *et al.*, 1998): The averages of the number of neighbors at a given distance are taken omitting those galaxies lying closer to the border than r. At large scales only a small fraction of the galaxies in the sample enters in the estimation, increasing the variance. To make full use of the surveyed galaxies, the estimator has to incorporate an edge-correction. The most widely used estimators in cosmology are the Davis and Peebles estimator (Davis and Peebles, 1983), the Hamilton estimator (Hamilton, 1993b), and the Landy and Szalay estimator (Landy and Szalay, 1993). Here we provide their formulae when applied to a complete galaxy sample in a given volume with N objects. A Poisson catalog, a binomial process with $N_{\rm rd}$ points, has to be generated within the same boundaries.

$$\widehat{\xi}_{\rm DP}(r) = \frac{N_{\rm rd}}{N}\frac{DD(r)}{DR(r)} - 1, \tag{21}$$

$$\widehat{\xi}_{\rm HAM}(r) = \frac{DD(r)\cdot RR(r)}{[DR(r)]^2} - 1, \tag{22}$$

$$\widehat{\xi}_{\rm LS}(r) = 1 + \left(\frac{N_{\rm rd}}{N}\right)^2 \frac{DD(r)}{RR(r)} - 2\frac{N_{\rm rd}}{N}\frac{DR(r)}{RR(r)}. \tag{23}$$

where $DD(r)$ is the number of pairs of galaxies with separation within the interval $[r - dr/2, r + dr/2$, $DR(r)$ is the number of pairs between a galaxy and a point of the Poisson catalog, and $RR(r)$ is the number of pairs with separation in the same interval in the Poisson catalog. At large scales the performance of the Hamilton and Landy and Szalay estimators has been proved to be better (Kerscher *et al.*, 2000; Pons-Bordería *et al.*, 1999).

3. Recent determinations of the correlation function

Earlier estimates of the pairwise galaxy correlation function were obtained from shallow samples, and one could suspect that they were not finding the true correlation function. The first sample deep enough to get close to solving that problem was the Las Campanas Redshift Survey (LCRS). The two-point correlation function for LCRS was determined by Tucker *et al.* (1997) and by Jing *et al.* (1998) (see Fig. 10). Jing *el al.* get slightly smaller values for the correlation length ($r_0 = 5.1h^{-1}$Mpc) than Tucker *et al.* ($r_0 = 6.3h^{-1}$Mpc). When making comparisons, it is necessary to take care that the length scales have been interpreted in the same underlying cosmological model. Older papers tend to set $\Lambda = 0$ whereas more recent papers are often phrased in terms of a flat-Λ plus cold dark matter cosmology.

Analyzing data from the first batch of the SSDS, Zehavi *et al.* (2002) analyse 29300 galaxies covering a 690 square degree region of sky, made up of a number of long narrow segments (2.5 - 5 degrees). They arrive at an average real-space correlation function of

$$\xi(r) = \left(\frac{r}{6.1 \pm 0.2 h^{-1}\mathrm{Mpc}} \right)^{-1.75 \pm 0.03} \qquad (24)$$

for $0.1h^{-1}$ Mpc $< r <$ 16 h^{-1} Mpc. This comes close to the LCRS result of Tucker *et al.* (1997). More recently, the same group (Zehavi *et al.*, 2003) has updated the result, using a more complete sample with 118,149 galaxies (see Fig. 11), and the best power-law fit is

$$\xi(r) = \left(\frac{r}{5.77 h^{-1}\mathrm{Mpc}} \right)^{-1.80} \qquad (25)$$

This is a remarkable scaling law covering some 3 orders of magnitude in distance. The smallest scale measured ($100h^{-1}$ kpc) is barely larger than a typical galaxy. Interestingly, this lower scale is set, in the Zehavi *et al.* (2002) analysis, by the requirement that, at the outer limit of the survey (corresponding to a radial velocity of 39,000 km s^{-1}), pairs of galaxies should be no closer than can be reached by two neighboring fibers on the multifiber system. There would be some interest in looking at nearer galaxies and tracing the correlation function to even smaller scales to see whether the old and remarkable extrapolation of Gott and Turner (1979)[6] is valid in this newer data set (see also Infante *et al.* (2002)). The largest distance ($16h^{-1}$ Mpc) is larger than the size of a great cluster. It should be emphasized that this is a real space correlation function: the finger-of-god effects have been filtered.

There is a substantial luminosity effect seen in the scale length. The absolute magnitude $M_\star$ of the "knee" of the Schechter galaxy luminosity function (Schechter, 1976) is taken as a reference point (being a "typical" galaxy luminosity, whatever that means). For galaxies with absolute magnitudes centered on $M_\star - 1.5$ the scale length is $r_0 \approx 7.4h^{-1}$ Mpc. For samples centered on $M_\star$ the scale length is $r_0 \approx 6.3h^{-1}$ Mpc. And for samples centered on $M_\star + 1.5$ the scale length is $r_0 \approx 4.7h^{-1}$ Mpc. The slope for these samples is essentially the same. A similar strong dependence of the correlation function on the color, morphology, and redshift of galaxies was found before, in the Canadian Network for Observational Cosmology Field Galaxy Redshift Survey (CNOC2) by Shepherd *et al.* (2001).

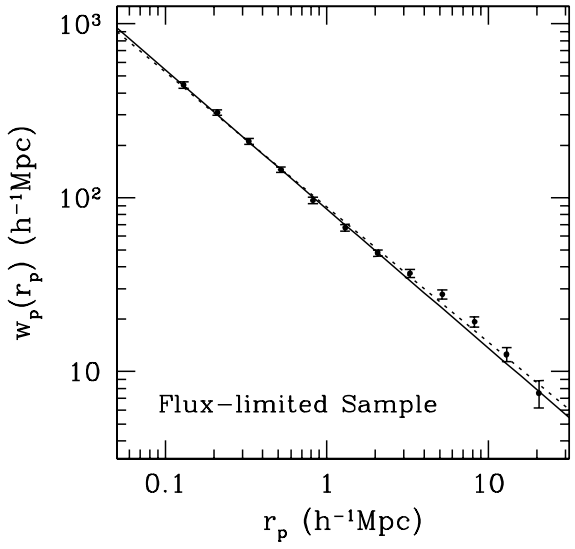

FIG. 11 The (projected) real space two point-correlation function of the SSDS data from Zehavi *et al.* (2003). The two straight lines show different fits corresponding to different weighting schemes

The angular correlation function for the SDSS (Connolly *et al.*, 2002) is independent of redshift distortions and agrees well with the value inferred from the redshift survey. This encourages one to believe that the redshift corrections are being handled effectively.

However, the latest careful analysis of the (almost) full 2dF survey (Hawkins *et al.*, 2003) gives the correlation length $r_0 = 5.05h^{-1}$Mpc, substantially smaller than the SDSS result. Hawkins *et al.* (2003) ascribe this to the different galaxy content of the two surveys: the SDSS is a red-magnitude selected survey and the 2dFGRS is a blue magnitude selected survey.

4. Correlation dimension

Recently, many authors have measured the correlation dimension of the galaxy distribution at different scales using all available redshift catalogs. Wu *et al.* (1999) and Kurokawa *et al.* (2001) summarized these results in a table. A more completed and updated version of a similar table, includ-

[6] Gott and Turner estimated the small-scale end of the correlation function down to a scale of $30h^{-1}$ kpc from the distribution of projected distances between isolated galaxy pairs (double galaxies). As strange as it may seem, this correlation function fitted neatly the general galaxy correlation function.

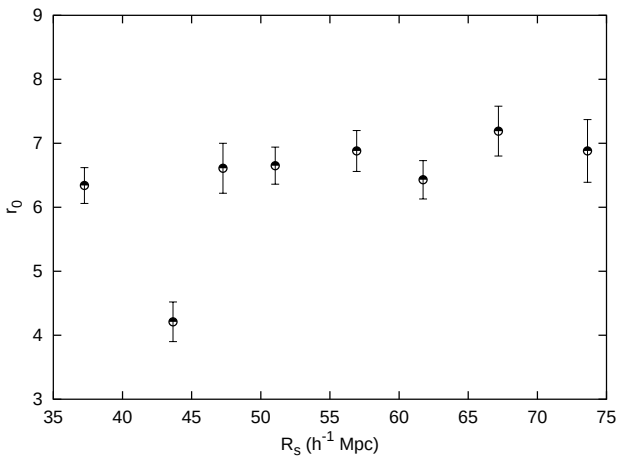

FIG. 12 The correlation length as a function of the sample depth for the CfA-II catalog, from Martínez *et al.* (2001). The observed *plateau* argues against the fractal interpretation of the galaxy distribution.

ing more references and new catalogs is presented here (see Table I). The estimates of the correlation dimension have been performed using different methods depending on the authors' preferences. It is worthwhile to mention the elegant technique introduced by Amendola and Palladino (1999) based on radial cells that maximizes the scale at which the minus-estimator can be applied. The table shows unambiguously that the correlation dimension is a scale dependent quantity, increasing gradually from values $D_2 \simeq 2$ for scales less than $20 - 30h^{-1}$ Mpc (and even larger values of D_2 in IRAS based redshift surveys) to values approaching $D_2 \simeq 3$ for larger scales.

5. Correlation length as a function of sample depth

The first indication that correlation length might depend on the sample depth was found in the CfA-I data (Einasto *et al.*, 1986). The correlation length increased, when deeper samples were chosen. Although the authors explained the effect by the specific geometry of the mass distribution in shallow samples, this paper motivated the early campaign to explain the galaxy distribution as fractals (Calzetti *et al.*, 1988; Pietronero, 1987), because for a fractal r_0 increases proportionally with the sample depth (Coleman and Pietronero, 1992; Guzzo, 1997). The Ruffini group realized from the beginning that fractal scaling cannot extend to large scales and started to look for crossover to homogeneity (Calzetti *et al.*, 1991), but the Pietronero group has continued the fractal war until now, fighting for an all-fractal universe. Their stand is summarized in Sylos Labini *et al.* (1998).

The deep samples now at our disposal have solved this problem once and for all — the galaxy correlation functions may depend on their intrinsic properties (luminosity, morphology, etc.), but not on the sample size (Kerscher, 2003; Martínez *et al.*, 2001). As an example, Fig. 12 shows the results of a recent study.

C. Galaxy-galaxy and cluster-cluster correlations

Having re-discovered the power of the two-point correlation function as a tool for measuring clustering, it was evident that the Princeton group would go on to analyze every available catalog of extragalactic objects they could lay their hands on. One of these catalogs was the Abell catalog of rich galaxy clusters identified on the Palomar Sky Survey (Hauser and Peebles, 1973). The technique used was power spectrum analysis since it was felt this would give a better method of dealing with the incomplete sky coverage.

It came as somewhat of a surprise to discover (a) that these Abell clusters were themselves clustered and (b) that, on a given scale, they were more clustered than the galaxies. The former was a surprise because serious doubts had previously been expressed about the reality of superclustering. Here was direct evidence that clusters were likely to be found in pairs and even in groups. The latter was a surprise because it had been (naively) expected that clusters identified from a set of points would necessarily have the same correlation function as the set itself. The galaxy clusters were themselves clustered on scales where the galaxy-galaxy correlation was so small as to be immeasurable.

Both the galaxy and cluster correlation functions are approximately power laws $\xi(r) = (r/r_0)^{-\gamma}$ with the same exponent $\gamma \approx 1.8$, but the correlation amplitudes for clusters are much larger than those for galaxies.

There is a simple reason why the cluster-cluster correlation function might have an amplitude exceeding the galaxy-galaxy correlation function amplitude: it arises because of the way clusters are identified as regions where groups of points have a substantially higher than average density. Such regions contain most of the close pairs that go into defining the value of the galaxy-galaxy correlation function. Moreover, eliminating the points which are not in such clusters biases the expected number of pairs that would have been found had this been a Poisson distribution containing the same number of points. The boost in the value of the correlation function achieved from such censorship depends directly on the volume of space occupied by these clusters.

This entirely obvious point was made in a preprint by Jones and Jones (1985): the paper was never published. As with many useful ideas, it became common knowledge and moved into the realm of folklore.

There remained some important questions:

a: Does the Abell catalog provide a sufficiently good sample for this purpose: is it free from systematic biases that may prejudice the result? Abell identified clusters by eye, a procedure which would lack the objectivity of an automatic plate scanning machine.

b: If in the cluster sample we reject the least impressive ones, would this change the correlation function? This corresponds to selection by cluster richness.

c: How would changing the selection threshold affect the correlation function? This is not quite the same as selecting by cluster richness: less rich clusters are still

TABLE I The correlation dimension estimated on different redshift surveys at different scale ranges.

Reference	Sample	Range of scales (h^{-1} Mpc)	D_2
Martínez and Jones, 1990	CfA-I	3-10	$1.15 - 1.40$
Lemson and Sanders, 1991	CfA-I	$1 - 30$	2
Domínguez-Tenreiro *et al.*, 1994	CfA-I	$1.5 - 25$	2
Kurokawa *et al.*, 1999	CfA-II	$7 - 27$	1.89 ± 0.06
Guzzo *et al.*, 1991	Perseus-Pisces	$1 - 3.5$	1.25 ± 0.10
	Perseus-Pisces	$3.5 - 27$	2.21 ± 0.06
	Perseus-Pisces	$27 - 70$	$\simeq 3$
Martínez *et al.*, 1998	Perseus-Pisces	$1 - 20$	$1.8 - 2.3$
Martínez and Coles, 1994	QDOT	$1 - 10$	2.25
	QDOT	$10 - 50$	2.77
Martínez *et al.*, 1998	Stromlo-APM	$30 - 60$	$2.7 - 2.9$
Hatton, 1999	Stromlo-APM	$12 - 55$	2.76
Amendola and Palladino, 1999	Las Campanas	$\leq 20 - 30$	2
	Las Campanas	> 30	$\rightarrow 3$
Kurokawa *et al.*, 2001	Las Campanas	$5 - 32$	1.96 ± 0.05
	Las Campanas	$32 - 63$	$\simeq 3$
Pan and Coles, 2000	PSCz	< 10	2.16
	PSCz	$10 - 30$	2.71
	PSCz	$30 - 400$	2.99

included, though they would appear as smaller objects on increasing the discrimination threshold.

d: If clusters were selected other than by virtue of their contrast with the background, eg: from identifying clusters in an X-Ray survey, would we still see enhanced clustering?

e: What does the galaxy-cluster cross correlation tell us?

It was well known that there were systematic biases in the Abell Catalog. The subsample of low richness clusters was incomplete, and the more distant clusters were systematically richer than than nearby counterparts. This was not in itself enough to remove the "discrepancy" between the galaxy-galaxy correlation function and the cluster-cluster correlation function, but it might prejudice conclusion about richness dependence of the discrepancy.

It was not until 1992 that a sufficiently good alternative to the Abell Catalog became available: this was the APM cluster catalog (Dalton *et al.*, 1992, 1997) derived from the Cambridge APM Galaxy Survey (Automatic Plate Measuring Machine) of UK Schmidt Telescope plates. Now we await results from the large 2dF and SDSS redshift catalogs which have already provided detailed information about the galaxy-galaxy correlation function.

1. Analysis of recent catalogs

Currently the best data on galaxy cluster clustering comes from redshift surveys of clusters identified in machine generated galaxy catalogs and of clusters observed in X-Ray surveys. The 2dF and SDSS surveys will undoubtedly settle this matter once and for all since they contain a large number of clusters that can be selected on the basis of redshift. However, it is already apparent (as in the Shepherd *et al.* (2001) study of the CNOC2 sample, for Zehavi *et al.* (2002) study of the Early SDSS Data, and for Madgwick *et al.* (2003); Norberg *et al.* (2001) correlation analysis of the 2dFGRS) that talking about *the* galaxy-galaxy correlation function is somewhat of an oversimplification in the first place: the galaxy-galaxy correlation depends strongly on the absolute magnitude, galaxy colour and galaxy spectral type. Galaxies are clearly not unbiased tracers of the underlying mass distribution.

In automated cluster searching, clusters are generally discovered via a nearest-neighbour, friends-of-friends, type of analysis. They are discovered by virtue of their central concentration and so catalogs contain clusters that are defined in terms of a "distance to your nearest neighbour" threshold length. If the threshold length is increased the catalog contains more clusters: the number of poorer, less centrally dense, clusters increases. It is not a priori obvious how the mean density of galaxies within a cluster so found relates to its central density: there will clearly be a correlation. It might well be that selecting clusters by virtue of their mean galaxy density rather than their peak density would yield different catalogs and lead to different conclusions about the systematics of cluster-cluster clustering.

2. Theoretical expectations

It is easier to build theoretical (analytic) models based on selection by mean cluster density, ie: clusters selected via a density threshold, than it is to build models based on clusters selected by peak density. The latter requires an understanding of how the cluster dynamics works to produce the density profile of the galaxy distribution. This may contribute to some of the confusion that exists when looking for trends in the clustering of clusters.

The earlier theoretical models (Bahcall and West, 1992; Jones and Jones, 1985; Kaiser, 1984) for the clustering of clusters were based on threshold selection. The same is true of more recent hierarchical models based on multifractal models for the distribution of galaxies (Martínez *et al.*, 1990; Paredes *et al.*, 1995). Most of the conclusions about superclustering in which the clusters are defined via the peak density excursion comes from N-body simulations of various sizes and sophistication (Bahcall and Cen, 1992; Colberg *et al.*, 1998; Croft and Efstathiou, 1994).

Since clusters found in X-ray surveys are found by virtue of their gas temperature, that is total potential, these surveys should agree rather well with the conclusions based on N-body experiments.

3. Richness dependence of the correlation length

The seminal paper on the effect of cluster richness on the cluster–cluster correlation function was that of Szalay and Schramm (1985). They suggested that the scaling length for clustering should itself depend on the cluster density. Which cluster density, peak or mean, was never stated.

The formula for the cluster two-point correlation function $\xi(r;\nu)$ is usually written as (Kaiser, 1984):

$$\xi(r;\nu) = \frac{\nu^2}{\sigma^2}\xi(r), \tag{26}$$

where ν is the height of the peaks in units of the rms error σ of the galaxy density field, and $\xi(r)$ is the correlation function of the galaxy field[7].

The empirical determination of the the cluster–cluster correlation function, $\xi_{cc}(r)$, is much more uncertain than the galaxy–galaxy correlation function, $\xi_{gg}(r)$. The selection effects associated with the cluster identification method (Eke *et al.*, 1996) are the major source for this uncertainty. The possible dependence of clustering properties on cluster richness makes the issue still more difficult. Nevertheless $\xi_{cc}(r)$ is usually fitted to a power law

$$\xi_{cc} = \left(\frac{r}{r_c}\right)^{\gamma_c}. \tag{27}$$

Eq. 26 holds if $\gamma_c = \gamma$, where γ is the exponent of the power-law galaxy–galaxy correlation function. As already mentioned, this seems to be the case, see for example in Fig. 13, the remarkable agreement between the slopes of the correlation function of the REFLEX cluster catalog and the Las Campanas galaxy redshift survey (Borgani and Guzzo, 2001; Guzzo, 2002). Nevertheless, depending on the analyzed cluster sample and cluster identification procedure, the scatter of the reported values for the slope of the correlation function is very high with $\gamma_c = 1.6$ to 2.5. For

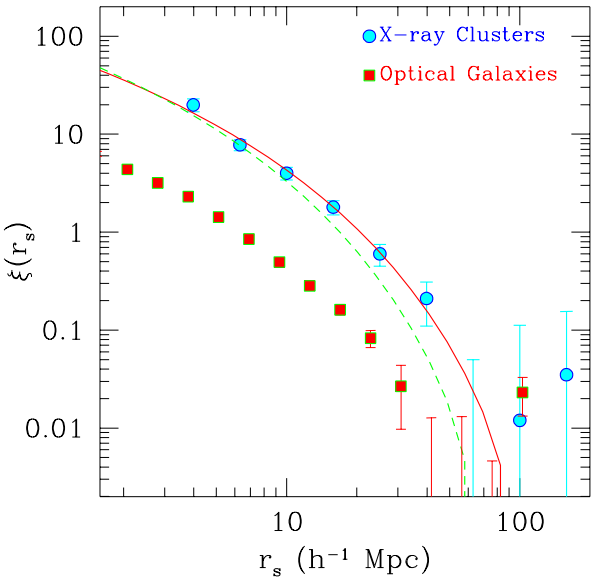

FIG. 13 The two-point correlation function for the X-ray selected clusters from the REFLEX survey (circles) and for the Las Campanas galaxy redshift survey (squares). The solid and dashed lines are the expected results for an X-ray similar survey in a LCDM model with different values for the cosmological parameters, from Borgani and Guzzo (2001).

the correlation length the values go from $13h^{-1}$ Mpc to $40h^{-1}$ Mpc (Bahcall and West, 1992; Borgani and Guzzo, 2001; Dalton *et al.*, 1994; Nichol *et al.*, 1992; Postman *et al.*, 1992). Fig. 14 illustrates this variability displaying the differences between the correlation function of the Abell and APM cluster samples.

Rich clusters have many members and are rare, therefore the distance between then $d_c = n_c^{-1/3}$ is larger. Bahcall and West (1992) derived a linear relation between the cluster correlation length r_c and the mean intercluster separation d_c, $r_c = 0.4d_c$ from power-law fits (constrained to have a fixed value of $\gamma_c = 1.8$) to correlation functions calculated on cluster samples with different richness. Fig. 15 shows that this relation is not confirmed by the new data. In fact, at large values of d_c the relation must level off, and a weaker dependence of r_c versus d_c agrees better with the observations, for example $r_c = 2.6\sqrt{d_c}$ as shown in the figure (Bahcall *et al.*, 2003).

Since r_c and γ_c are not independent, the slope is usually constrained to a fixed value $\gamma_c = 1.8$. Dependence of γ_c on cluster richness has been proposed (Martínez *et al.*, 1995), although this dependence is better parametrized by the correlation dimension — the exponent of the power law fitting the correlation integral $N(r) = Ar^{D_2}$ (see Eq. 16). Multiscaling is the term used for scaling laws in which D_2 displays a slowly varying behavior with the density threshold that characterizes the richness of clusters. The higher the threshold, the richer the clusters, and the smaller the value of D_2. Within the multiscaling framework, the relation r_0 versus d_c gets a more

[7] As the correlation functions and σ are defined for the density contrast $\delta = (\rho - \bar{\rho})/\bar{\rho}$, all quantities in (26) are dimensionless; there is no dimensionality conflict.

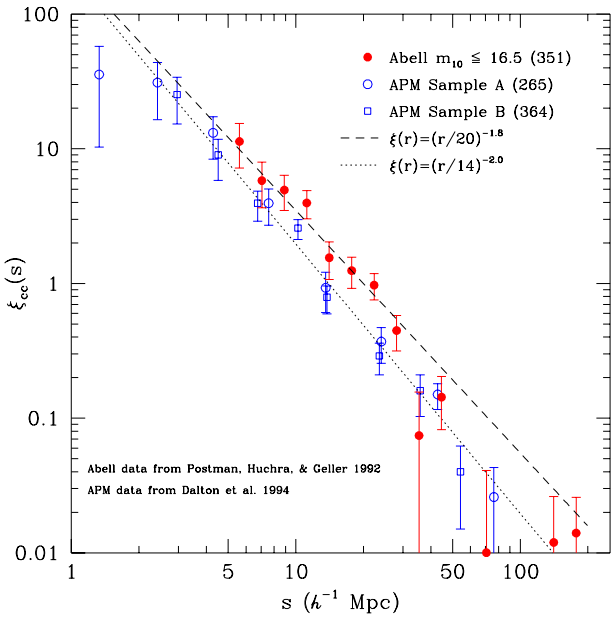

FIG. 14 The two-point correlation functions for the Abell clusters and two subsamples of the APM survey. The best power-law fits are shown in the plot, from Postman (1999).

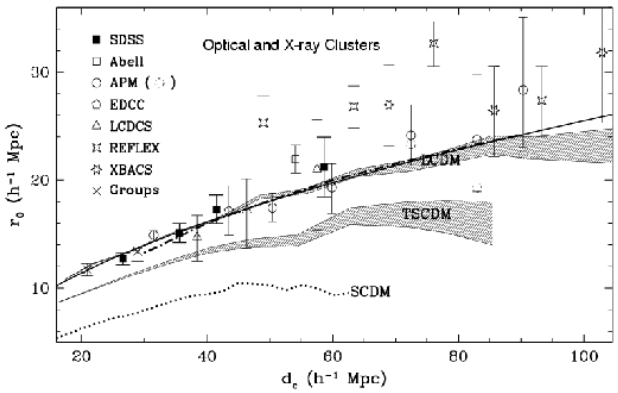

FIG. 15 The correlation length of different cluster samples as a function of the intercluster distance. The solid line shows the relation $r_c = 2.6\sqrt{d_c}$ that fits well the observations and the LCDM model, from Bahcall *et al.* (2003).

complicated form flattening for large values of d_c as the observations confirm (Martínez *et al.*, 1995).

D. The pairwise velocity dispersion

The pairwise velocity dispersion of galaxies is a measure of the temperature of the "gas" of galaxies. By energy conservation, the kinetic energy of this gas has to be balanced by its gravitational energy, which depends mainly on the mean mass density of the Universe. Thus, measuring the pairwise velocity dispersion gives us a handle on the density. This is, however, more easily said than done since we measure only the radial component of the velocity, and that is biased by larger density inhomogeneities than a linear theory can handle.

The following short argument shows how the velocity dispersion relates to the fluctuations in the density field. The non-Hubble component of a galaxy velocity through the Universe, (its *peculiar* velocity), is due to the acceleration caused by clumps in the matter distribution. This is easy to estimate during the phase of linear evolution of cosmic structure since linear perturbation theory applies.

A particle that has experienced a peculiar acceleration g_p for a time t would have acquired a peculiar velocity $v_p \sim g_p t$. If this acceleration is due to a mass fluctuation δM at distance r, we have

$$g_p = G\delta M/r^2 = (4\pi/3)G\delta\rho r = 0.5\Omega_0 H_0 v_H \qquad (28)$$

which leads to

$$v_p/v_H \simeq (1/3)f(\Omega)\delta, \quad f(\Omega) = (3/2)H_0 t \simeq \Omega^{0.6}. \qquad (29)$$

For a more general approximation including the cosmological constant see Lahav *et al.* (1991). As one can see the ratio of the peculiar to Hubble velocity is the quantity that gives a direct measure of the amplitude of primordial density fluctuations on a given scale for a given value of Ω. If we have a scaling law for the density fluctuations we should also see a scaling law in the peculiar velocity field.

A more detailed calculation, still using linear theory, gives a direct relation between the *rms* amplitude of the peculiar velocity and the power spectrum of primordial density fluctuations (Strauss and Willick, 1995):

$$\langle v_p(R)^2\rangle = \frac{H_0^2 f^2}{2\pi^2}\int \mathcal{P}(k)\widetilde{W}^2(kR)dk \qquad (30)$$

where $\widetilde{W}(kR)$ is the Fourier transform of a spherical window function of radius R, $W(R)$. This equation also works quite well for rather high δ, well beyond the linear regime. The main problem then becomes dealing with the redshift distortion of the observed velocity field.

This equation, however, contains information only about the *rms* magnitude of v_p on a given scale. More information about peculiar motions in different cosmological scenarios can be obtained from other types of the velocity correlation functions that can be estimated from data sets.

As direct data on peculiar velocities of galaxies are hard to obtain, the pairwise galaxy velocity dispersion is measured from ordinary redshift surveys by modelling its effect on the redshift space correlation function. This modelling is not very certain, as it depends on the choice both of the adopted mean streaming velocity model and of the model for the pairwise velocity distribution itself. The latter is usually modelled as an exponential distribution (Peebles, 1980).

The first determination of the pairwise velocity dispersion σ_{12}^2 was made by Davis and Peebles (1983), who found $\sigma_{12} \approx 340\,\mathrm{km\,s^{-1}}$. Subsequent determination from the IRAS data (Fisher, 1995; Fisher *et al.*, 1994) gave a similar value ($\sigma_{12} \approx 317\,\mathrm{km\,s^{-1}}$). These values were much lower than those predicted for the Standard Cold Dark Matter (SCDM)

model ($\sigma_{12} \approx 1000\,\mathrm{km\,s^{-1}}$), and served as an argument for discarding the model.

Later determinations have given larger values for this dispersion: the estimates of Jing *et al.* (1998); Marzke *et al.* (1995); Zehavi *et al.* (2002) all converge at the value $\sigma_{12} \approx$ 550–600 $\mathrm{km\,s^{-1}}$; not enough for the SCDM model, but in concordance with the present standard, the ΛCDM model.

In the stable clustering model the pairwise velocity dispersion should scale with pair distance as $r^{0.2}$; this scaling has not been observed. Also, it is well known that the value of σ_{12} is sensitive to the presence of rich clusters in the sample. Davis *et al.* (1997) and Landy *et al.* (1998) propose alternative schemes for estimating the pairwise velocity dispersion, which again lead to small values of σ_{12}.

The galaxy velocity field is also rather inhomogeneous; a well-known fact is the extreme coldness of the flow in our neighbourhood, out to $5h^{-1}$Mpc, where $\sigma_{12} = 60\,\mathrm{km\,s^{-1}}$ (Schlegel *et al.*, 1994).

E. Light does not trace mass

It has long been realized that there is a difference between the distribution of light in the Universe and the distribution of mass. The first clues came with the apparent systematic increase of mass-to-light ratios with scale determined from galaxies, binary galaxies, groups and clusters of galaxies: this was later made more explicit by Einasto *et al.* (1974), Joeveer and Einasto (1978) and Ostriker *et al.* (1974). It was also known that galaxy morphology is related to the clustering environment (Abell, 1958; Davis and Geller, 1976; Dressler, 1980; Einasto *et al.*, 1980; Guzzo *et al.*, 1997; Hubble, 1936; Zwicky, 1937).

The recognition that clustering depends on galaxy luminosity is more recent (Benoist *et al.*, 1999; Domínguez-Tenreiro and Martínez, 1989; Hamilton, 1988; Kerscher, 2003; Loveday *et al.*, 1995; Martínez *et al.*, 1993; White *et al.*, 1988; Willmer *et al.*, 1998).

It is not difficult to understand why this should be so. We may be even surprised that the results were in any way surprising! There was early work of Bahcall and Soneira (1983); Bardeen *et al.* (1986); Melott and Fry (1986). However, it has not been easy to model these luminosity— and type-dependent phenomena since we have only the barest understanding of the galaxy formation process and it is probably fair to say that our knowledge of what causes galaxies to have vastly different morphologies is still rather incomplete.

The recent advances in augmenting N-body simulations with semi-analytic models and computational hydrodynamics is promising, though at a relatively early stage (Benson *et al.*, 2000; Blanton *et al.*, 1999; Cen and Ostriker, 1992; Colín *et al.*, 1999; Katz *et al.*, 1992; Kauffmann *et al.*, 1999; Pearce *et al.*, 1999; White *et al.*, 2001; Yoshikawa *et al.*, 2001). Modelling the formation of individual galaxies shows just how many physical processes must be taken into account, quite apart from trying to fold in our ignorance of the star formation process (and that is what gives rise to the luminosity). A brave attempt is exemplified by the paper of Sommer-Larsen *et al.* (2003).

1. Mass distribution and galaxy distribution: biasing

The concept of biasing was introduced by Kaiser (1984) in order to explain the observed relation between the correlation functions of galaxies and galaxy clusters. Using the high-peak approximation to a Gaussian density field, he obtained a formula (26) showing that the two correlation functions were proportional.

The same idea was later applied to galaxy distributions: as different types of galaxies have different clustering properties, they cannot all follow directly the overall density field. Thus we normalize the correlations by writing

$$\sigma^2_{gal} = b^2 \sigma^2_{total}, \tag{31}$$

(note that $\sigma^2 = \xi(0)$), and call b the bias factor. As baryonic matter comprises about four per cent of the total matter plus energy content of the universe, we can also say that the above relation connects the galaxy and dark matter distributions.

Bias cannot be measured directly, and indirect observational determinations of bias values have not yet converged to a single value for a given type of galaxies. Moreover, Dekel and Lahav (1999) showed that bias is, in general, nonlinear and stochastic. And later determinations have found that bias is also scale-dependent (Hamilton *et al.*, 2000). Such bias can easily destroy scaling relations that could be inherent in the matter distribution.

2. Mass and light fluctuations

An alternative measure of the scale dependence of clustering is to plot the variance of the mass or light density fluctuations on a variety of scales. This is little more than what Carpenter had done in the 1920's, and was first formalized by Peebles (1965) in his remarkable paper on galaxy formation. [8] It is relatively easy to calculate a density fluctuation spectrum: sample the density field in windows of different sizes, for each window size calculate the mean and variance of the contents of the window and plot the result. This works equally well in two or three dimensions. Some important technical questions arise: what to do at the boundaries and what the shape and profile of the window should be. By the profile it is meant what weight is attached to an object falling at a given location in the window. The "top hat" profile counts a weight of one if the object is in the window and zero outside: this is the simplest choice, though not a particularly good one. Fuzzy edged

[8] Several things are remarkable about Peebles' 1965 paper. It was Peebles' first paper on galaxy formation and its submission to the Astrophysical Journal preceded the the announcement of the discovery of the microwave background. In that paper we see the entire roadmap for the following decades of galaxy formation theory, albeit in terms of initial isothermal fluctuations.

windows are to be preferred since they reduce the effects of shot noise.

This process is analogous to two other methods of analyzing a density field: counts in cells and wavelet analysis. Counts in cells statistics do precisely what has just been described, using various coverings of the data set, and are most often hard-edged. The wavelet analysis does the same, but the choice of analyzing wavelet determines how "hard" the sampling volume is. Simple Haar wavelets are a bad choice since they too are hard-edged, but there are many fine alternatives. This an an area which requires more research since wavelets are particularly good at sniffing out scaling relationships.

The density fluctuation spectrum is in some sense a halfway house towards the power spectrum: the variance of the mass fluctuations are referred to a physical variable, mass scale, rather than the k-space wavenumber (which is itself an inverse length scale). The problem with the mass spectrum is that its amplitudes are correlated and depend on the adopted mass profile filter; the conventional power spectrum (spectral density) has independent amplitudes as it will be explained in Sect. VI.C.

VI. FURTHER CLUSTERING MEASURES

A. Higher order correlation functions

The two-point correlation function is not a unique descriptor of clustering, it is merely the first of an infinite hierarchy of such descriptors describing the galaxy distribution of galaxies taken N at a time. Two quite different distributions can have the same two-point correlation function. In particular, the fact that a point distribution generated by any random walk (e.g., as a Lévy flight as proposed by Mandelbrot (1975) has the correct two-point correlation function does not mean much unless other statistical measures of clustering are tested.

The present day galaxy distribution is manifestly not a Gaussian random process: there is, for example, no symmetry about the mean density. This fact alone tells us that there is more to galaxy clustering than the two-point correlation function.

So what kind of descriptors should we look for? Generalizations of the two-point functions to 3-, 4- and higher order functions are certainly possible, but they are difficult to calculate and not particularly edifying. However, they do the job of providing some of the needed extra information and through such constructs as the BBGKY hierarchy[9] they do relate to the underlying physics of the clustering process. We shall describe the observed scaling of the 3-point correlation function below.

One alternative is to go for different clustering models: anything but correlation functions. These may have the virtue of providing immediate gratification in terms of visualization of the process, but they are often difficult to relate to any kind of dynamical process.

If we knew all higher order correlation functions we would have a complete description of the galaxy clustering process. However, calculating an estimate of a two point function from a sample of N galaxies requires taking all pairs from the sample of N, while calculating a three point functions requires taking all triples from N. The amount of computation escalates rapidly and restrictions have to be imposed on what is actually being calculated.

Nevertheless, calculating restricted N-point functions may be useful: these functions may be related to one another and have interesting scale dependence. Gaztañaga (1992) has calculated restricted N-point functions and showed that these have power law behavior over the range of scales where they can be determined.

B. Three-point correlation functions

The simplest high-order correlation function is the 3-point correlation function $\zeta(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. It appears to be simply related to the two-point function through a Kirkwood-like relationship (see Peebles (1980)):

$$\begin{aligned}\zeta(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \zeta(r_{12}, r_{23}, r_{31}) = \\ = Q\left[\xi(r_{12}\xi(r_{23}) + \xi(r_{23})\xi(r_{31}) + \xi(r_{31})\xi(r_{12})\right],\end{aligned} \tag{32}$$

where $Q \approx 1$ is a constant, and the first equality is due to the usual assumption of homogeneity and isotropy. This scaling law is called "the hierarchical model" in cosmology, and it agrees rather well with observations. The full Kirkwood law(Ichimaru, 1992) would require an additional term on the right-hand side of Eq. (32), proportional to $\xi(r_{12})\xi(r_{23})\xi(r_{31})$.

As observations show (Meiksin *et al.*, 1992; Peebles, 1980, 1993), there is no intrinsic 3-point term, either Kirkwood type or more general. If this term were present the 3-point function would be enormous at small scales. Therefore it makes no contribution. The absence of such a 3-point term is probably a consequence of the fact that gravity is a two-body interaction and is the only force that plays a role in the clustering process.

C. The power spectrum

The power spectrum $P(k)$ is the description of clustering in terms of wavenumbers k that separates the effects of different scales. If $F(\mathbf{k})$ is the Fourier transform of a random field, then

$$\mathcal{P}(\mathbf{k}) = \mathrm{E}\left[F(\mathbf{k})\overline{F(\mathbf{k})}\right] \tag{33}$$

where E denotes the statistical expectation value.

The Fourier modes of a Gaussian random field (our basic model for the matter distribution in the universe at early times) are independent, and the only function that defines the field is

[9] The BBGKY hierarchy, (after Bogolyubov, Born, Green, Kirkwood and Yvon), is an infinite chain of equations adapted from plasma physics (Ichimaru, 1992) to describe self-gravitating non-linear clustering (see for example Fall and Severne (1976), Peebles (1980), and Saslaw (2000).)

the power spectrum. As the initial fluctuations from the inflation period are described naturally in terms of Fourier modes, the power spectrum is the best descriptor of the matter distribution for these times.

Inflationary models predict a power-law power spectrum, $P(k) \sim k^n$ (see Peebles and Ratra (2003) for a recent review), with the most popular exponent $n = 1$. This simple scaling is, however, broken, once the wavelength of a mode gets smaller than the horizon; interactions between matter, radiation and gravity deform the power spectrum in a computable, but complex manner (Eisenstein and Hu, 1998, 1999).

Nevertheless, if we restrict ourselves to a smaller scale interval (say, two orders of magnitude), the power spectrum remains close to a power law. For the scales of the observed structure the exponent of this power law is negative, ranging from $n = -1$ for larger scales to $n \geq -3$ for galaxy scales.

If we combine a scale-free power spectrum $P \sim k^n$ with a scale-free expansion law $a(t) \sim t^\alpha$ we should get a perfect scaling regime for evolution of structure. Unfortunately, this is not true, as there are two completely different regimes of evolution of gravitating structures: the linear regime, when every Fourier mode grows at the same rate, and the nonlinear regime, when we can assume that objects are virialized and their physical structure does not change. The latter assumption is called "stable clustering" (Peebles, 1974b).

The linear regime is characterized by small density amplitudes and large scales (small wavenumbers), the stable clustering regime has large density amplitudes and occurs at small scales (large wavenumbers). The scaling solution for the correlation function in the stable clustering regime was found by Peebles (1974b): $\xi(r) \sim r^{-\gamma}$, where $\gamma = (9 + 3n)/(5 + n)$. The first attempt to get a solution that would interpolate between the two regimes was made by Hamilton *et al.* (1991). For that they rescaled the distances r, assuming no shell crossing during evolution of objects, and found an empirical relation between the nonlinear and linear correlation functions, using N-body models. This is known as the HKLM scaling solution. Peacock and Dodds (1996) found a similar relation for power spectra. These results have been used frequently for likelihood search in large volumes of cosmological parameter space, which could not be covered by time-consuming N-body modelling.

However, nowadays it seems that the stable clustering hypothesis does not describe well either the observed structure, or present-day numerical simulations, mostly because of merging of objects in the later stages of evolution of structure. A scaling solution in terms of a nonlinearity wavenumber that does not assume stable clustering is described by Smith *et al.* (2003). Let us define the nonlinearity wavenumber k_{NL} by

$$\sigma^2(k_{NL}, a) \sim \int_0^{k_{NL}} P(k, a)\, k^2 dk = 1;$$

it separates the linear regime $k < k_{NL}$ from the nonlinear regime $k > k_{NL}$. One then expects the scaling solution to have the form

$$P(k, a) = F(k/k_{NL}).$$

As an example, for the Einstein-de Sitter cosmological model $a(t) \sim t^{2/3}$, the scale-free power spectrum can be written as $P(k, a) = a^2 k^n$, and the nonlinearity wavenumber $k_{NL} \sim a^{-2/(n+3)}$. Numerical experiments confirm that scaling solutions exist.

The latest real-space power spectrum of the SDSS survey (Tegmark *et al.*, 2004) shows clearly curvature, departing from a single power-law, providing, as the authors say, "another nail into the coffin of the fractal universe hypothesis".

D. The bispectrum

The power spectrum (Eq. 33) is a quadratic descriptor of a random field: it contains information about the amplitudes of the Fourier components, but not about any phase relationships that might have evolved through nonlinear processes. The power spectrum characterizes fully a Gaussian field. Since the present-day high-amplitude fluctuating density field is not Gaussian (there cannot be any region with negative density), the power spectrum by itself is provides only a partial description. There are several ways of providing further information in Fourier space, one of which is to look at higher order correlations among Fourier components.

The next order descriptors are cubic, the three-point correlation function and its Fourier counterpart, the bispectrum. The bispectrum is the third moment of the Fourier amplitudes of a random field, depending on three wavenumbers. If we denote the Fourier amplitudes of a random field by $F(\mathbf{k})$, the bispectrum of the field is defined as

$$\mathcal{B}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \mathrm{E}\left[F(\mathbf{k}_1)F(\mathbf{k}_2)F(\mathbf{k}_3)\right],$$

where E denotes the statistical expectation value. For homogeneous random fields the bispectrum is non-zero only for closed triangles of vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ (see, e.g., Martínez and Saar (2002)). Consequently, for real-valued homogeneous random fields the bispectrum can be calculated by

$$\mathcal{B}(\mathbf{k}_1, \mathbf{k}_2) = \mathrm{E}\left[F(\mathbf{k}_1)F(\mathbf{k}_2)\overline{F(\mathbf{k}_1 + \mathbf{k}_2)}\right],$$

where the overline denotes conjugation. In the signal processing world the bispectrum is known as the bicoherence spectrum and it is used to measure the phase coherence among triples of spectral components that arises as a consequence of nonlinear wave coupling.

The hierarchical ansatz that we wrote for the three-point correlation function can be written also for the bi-spectrum:

$$\begin{aligned}\mathcal{B}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= Q\left[P(\mathbf{k}_1)P(\mathbf{k}_2) + P(\mathbf{k}_1)P(\mathbf{k}_3) + \right. \\ &+ \left. P(\mathbf{k}_2)P(\mathbf{k}_3)\right].\end{aligned}$$

A similar expression is predicted by perturbation theory (Fry, 1984b), but with different coefficients for every term.

It is not easy to determine the bi-spectrum from observations, as its argument space is large (the set of all triangles), and it is strongly modified by galaxy bias. The estimates so far have confirmed that the bispectrum follows approximately the predictions of the perturbation theory (Bernardeau *et al.*,

2002). As it depends on the bias parameters, it can be used to estimate galaxy bias. An example is provided by a recent study (Verde *et al.*, 2002) that found that the bi-spectrum of the 2dFGRS galaxies is compatible with no bias; these galaxies seem to faithfully trace the total matter distribution.

E. Fractal descriptors of clustering

None of the previous descriptors is motivated by the requirement that the galaxy distribution should, in some sense, be scale free, which might be expected on the grounds that the gravitational force which drives the clustering is scale free. What one would like to do is to generate a set of scaling indices that describes, say, the scaling of the moments of the galaxy counts distribution with cell size.

This was in a sense achieved by Gaztañaga (1992, 1994) when he determined the scaling laws of restricted N-point correlation functions. However, one might argue that the scaling of some high order correlation function has less immediate intuitive appeal than the scaling of the moments of cell counts.

There is a formalism for describing moments of cell counts that is commonly used when describing fractal point sets that was adopted as a clustering descriptor by Martínez *et al.* (1990). If it is possible to determine a set of such scaling indices we can turn the argument around and say that, over the range of scales where scaling is observed, the galaxy clustering can be represented by a fractal of a given type.

One should be aware that having a power law correlation function is not necessarily an indication of scale invariance! Conversely, the fractal description implies no particular underlying physical process: it is merely a statement of how moments of counts in cells behave as a function of cell size.

It is an interesting question of physics to formulate the physical process that might generate this distribution of scaling indices. This has been attempted by Jones (1999) for a simple nonlinear gravitational clustering model.

1. A cautionary word

There is a considerable difference between using the concept of fractal measure to describe a statistical process in some particular regime and saying "this distribution is such-and-such a fractal". There has been a set of papers observing scaling of a low order correlation function and jumping to the conclusions that (a) this scaling law holds at all scales (Sylos Labini *et al.*, 1998) and (b) this scaling law must be a consequence of some exotic phenomenon (Bak and Chen, 2001).

In the first case scaling laws can only be expected to hold over scales where nonlinear gravitational clustering has been at work. In the linear regime we merely see a reflection of the initial conditions: these have been revealed to us by the COBE experiment and by other microwave background anisotropy measurements. Indeed, it is a prediction of gravitational clustering theory that there should be a break in the scaling laws that reflect the transition between the linear and nonlinear regimes. We expect to see this as the transition to homogeneity that must occur on large scales.

There is no way out of this: the COBE results tell us that there will be large scales where the Universe is almost homogeneous.

In the second case there is absolutely no indication that anything more exotic than the force of gravitation is involved in the growth of clustering. On the contrary, the manifest successes of gravitational N-body experiments testify to the adequacy of gravity. We are not observing a critical phenomenon, nor are we on the verge of some marginal instability.

2. Structure from counts in cells

The first analyses of galaxy sky maps were done by dividing the sky into cells and counting the cell occupancy. As mentioned earlier, Bok (1934) and Mowbray (1938) established the non-uniformity of the galaxy distribution by counting galaxies in cells, and later Rubin (1954), Limber (1954), and Totsuji and Kihara (1969) used the Lick catalog published as cell counts in 1° cells. Peebles used the unpublished higher resolution data from the original notes of Shane and Wirtanen. Today, cell counts still provide an important mechanism for analysing point distributions since they are easier to deal with than the raw, unbinned, data.

3. Scaling properties of counts in cells

Whether we evolve a model numerically or make some analytic approximation it is necessary to characterize the clustering that develops in a quantitative manner. Conventionally, this is done by presenting the two-point correlation function $\xi(r)$ for the mass distribution. However, by itself this does not fully characterize the distribution of points. An important alternative is to look at the distribution of counts in cells as a function of cell size.

The relationship between the probability $P_N(V)$ of finding N galaxies in a sample volume V and the correlation functions of all orders was given by White (1979). The expression is not of any real use unless all correlation functions are known, or if there is a known relationship between them. Fry (1984a) and Balian and Schaeffer (1989a) computed the properties of the counts-in-cells distribution $P_N(V)$ on the hypothesis that the correlation functions of all orders form a particular scaling hierarchy in which the qth order correlation function $\xi^{(q)}$ based on a q-agon of points r_i scales as

$$\xi^{(q)}(r_1,\ldots,r_q) = \lambda^{\gamma(q-1)}\xi^{(q)}(\lambda r_1,\ldots,\lambda r_q). \qquad (34)$$

The hierarchy is described by a single scaling index γ. The data available at the time, the CfA survey, appeared to support both the form of $P_N(V)$ and this scaling hypothesis.

The special case of $P_0(V)$ is the "Void Probability Function" (VPF), that is the probability of a volume V containing zero galaxies. One can construct the probability distribution for having a void of a given size V in a distribution of galaxies

with given correlation properties (Fall *et al.*, 1976). It is given by White (1979)

$$P_0(n_0V) = e^{-n_0Va} \tag{35}$$

with

$$a = 1 + \sum_{i=2}^{\infty}(-n_0)^{i-1}\int w_i dV_1 \ldots dV_{i-1}. \tag{36}$$

Here n_0 is the mean space density of galaxies (or clusters), and w_i is the i-point correlation function of $(i-1)$ coordinates and is determined on linear scales by (among other things) the power spectrum of the primordial density fluctuations. For purely Gaussian fluctuations the sum in a is cut off beyond the second term. However, gravitational evolution destroys the Gaussian character of fluctuations and we are thus forced to make an *ansatz* regarding the relationship between second and higher order correlation functions either through BBGKY hierarchies (Fry, 1984b) or by pure guess.

White (1979) shows the relation between $P_0(V)$ and the cell count probabilities $P_N(V)$. Different clustering models have been proposed based on particular choices for the counts in cells (Borgani, 1993; Coles and Jones, 1991; Saslaw, 2000). A particular —and rather popular— way of analyzing the statistical properties of point sets is through the possible scaling of the moments of the counts in cells as it is explained in next section. Alternatively, one can consider the scaling of moments of counts of neighbors (Martínez and Coles, 1994).

4. Quantifying structure using multifractals

Given a model for the development of galaxy clustering we might like to predict the resulting distribution of cell counts since this provides a straightforward way of confronting the model with data.

Denote by $p(X;L)$ the probability that some quantity x takes on the value X when measured in a cell of size L. The distribution p can be characterized by its moments:

$$m_q(L) = \sum_{cells} p(X;L)X^q \tag{37}$$

If for some monotonic function $D(q)$ the moments scale with cell size L as

$$\sum_{cells} p(X;L)X^q \propto L^{(q-1)D(q)} \tag{38}$$

the point distribution is said to have scaling properties characterized by dimensions $D(q)$. The exponent is written in this way since the case $q = 1$ corresponds to the total number of particles in the sample volume, which is obviously independent of the cell size. The case $q = 2$ is related to the variance of the cell counts and to the two-point correlation function.

Eq. 38 does not describe arbitrary point distributions, but it does describe a large and important set of such distributions that have the property of multifractal scaling(Borgani, 1995). It has been argued that the observed galaxy distribution and the distribution of particles in an evolved N-Body simulation exhibit multifractal scaling.

There is a slightly different way of getting at the scaling exponents $D(q)$: via the *partition function* $Z(q,r)$. $Z(q,r)$ is related to the qth statistical moment of the distribution of points as viewed in cells of size r. Suppose the sample is drawn from a probability distribution $p(n;r)$ for finding n galaxies in a randomly chosen cell of scale r. The qth moment of the cell occupancy is defined as

$$m_q = \sum_{n=0}^{\infty} p(n;r)n^q \tag{39}$$

The partition function is then defined as

$$Z(q,r) = \frac{\mathcal{N}_r}{N^q}m_q. \tag{40}$$

If $n_i(r)$ denotes the occupancy of the ith cell in a partition of the sample space into $\mathcal{N}_r$ cells of scale r, the sample estimate for the partition function is

$$\bar{Z}(q,r) = \sum_{i=1}^{\mathcal{N}_r}\left[\frac{n_i(r)}{N}\right]^q \tag{41}$$

where N is the total number of points ($\sum n_i(r)$). Note that the ordering of the cells is not important and so the information on the relationship between neighbouring cells appears through the r-dependence of $Z(q,r)$.

The situation of interest is where, for all values of q, $Z(q,r)$ is found to scale as a power law in r:

$$Z(q,r) \propto r^{(q-1)D(q)} \propto r^{\tau(q)}, \tag{42}$$

where $\tau(q)$ is the scaling index of the partition function; see, e.g., Martínez *et al.* (1990). The function $D(q)$ defined in this way is a measure of some generalised dimension of order q for the distribution. This is simply a restatement of Eq. 38. Since Eq. 40 tell us $Z \propto m_q$, Eq. 37 and Eq. 39 are essentially the same.

$D(q)$ is the logarithmic slope of the moment generating function and of the partition function:

$$D(q) = \frac{1}{(q-1)}\frac{d\log m_q(r)}{d\log r} \tag{43}$$

$$= \frac{1}{(q-1)}\frac{d\log Z(q,r)}{d\log r}, \quad q \neq 1 \tag{44}$$

In computing $D(q)$ for a sample we would therefore expect to be able to see a reasonably straight line of data points in plot of either m_q or $Z(q,r)$ against r. Several aspects of finite-size data sample mitigate against this.

It should be noted that, technically, Eq. (42) needs be valid only in the limit $r \to 0$. This limit is impossible to take in the case of a discrete sample which is dominated by shot noise at distances much smaller than the mean particle separation. We can only ask for scaling over some well observed range. Likewise, we are unable to reliably compute (41) for large q since at large values of q the sum is dominated by whatever happens to be the single largest cluster of points in the sample.

5. Intermittency

An important feature of many statistical distributions is the phenomenon known as *intermittency*. Mathematically this describes a situation where the higher moments of the *spatial* distribution of some quantity dominate over the lower moments in a special way: there is an anomalous ratio between successive statistical moments as compared with a Gaussian process. The physical manifestation of this is that the quantity becomes spatially localised.

It is important to realize that, although we traditionally characterize the galaxy distribution via its two-point and three-point correlation functions, these have little or nothing to do with the visual appearance of the clustering pattern: voids, walls and filaments. These macroscopic features are manifestations of the fact that the higher order moments of the density distribution are dominant: the statistical distribution of galaxies is intermittent.

Intermittency can be quantified through a simple non-dimensional function involving higher order statistical moments of the distribution. Consider some random function of position $\psi(\mathbf{x})$ having a non-zero mean and a statistical distribution whose moments $\langle\psi^q\rangle$ are known . The *intermittency exponent* μ_q is defined in terms of the scaling properties of the moments by

$$\frac{\langle\psi^q\rangle}{\langle\psi\rangle^q} \sim \left(\frac{L}{l}\right)^{\mu_q}, \tag{45}$$

where l is some fiducial length scale. The *spatial* intermittency pattern is characterized by the q dependence of this ratio of moments. It is well known that a quadratic $q-$ dependence of μ_q corresponds to a lognormal distribution of ψ (eg. Jones et al., 1993).

Notice that $\langle\psi^q\rangle$ is simply the moment generating function for the process $\psi(\mathbf{x})$, and so the property of intermittency is a feature of the underlying statistics.

The assumption that the individual moments scale as per Eq. (38) guarantees the existence of μ_q and in this case we have

$$\mu_q = (q-1)D(q). \tag{46}$$

Since the quantity $\langle\psi^q\rangle$ for $q = 1$ has no scale dependence (it is the mean value for the field), Eqs. (45) and (46) provide the scaling law of the moments in the case of multifractal scaling:

$$\langle\psi^q\rangle \propto l^{(q-1)D(q)}, \tag{47}$$

μ_q is the standard notation for the intermittency exponent. μ_q is also called $\tau(q)$ in the multifractal literature; as in Eq. (42).

6. Multifractality

People are generally familiar with the notion of simple scaling in which a function of one variable is independent of the scale of the variable. A power law is the prototypical example: if $n(r) \propto r^\alpha$ then rescaling $r \to s = \lambda r$ recovers the same power law behavior, $n(r) \propto s^\alpha$. Only the amplitude and scale of the function have changed, the shape is the same.

This kind of scaling can be expressed mathematically in a way that is particularly relevant to the current discussion. Suppose that $p(X, L)$ is the probability of measuring a value X for some property of a system when the sample volume has been binned into cells of size L. Then the property X is said to exhibit "simple or finite scaling" when for some constants β and ν

$$p(X, L) = L^{-\beta} g\left(\frac{X}{L^\nu}\right). \tag{48}$$

for some function $g(x)$. In the jargon of fractals we say that the quantity X is distributed on a fractal with a single scaling index.

Following Kadanoff *et al.* (1989) we can define a more complicated kind of scaling, *multifractal scaling*, in which we have

$$\frac{\log p(X, L)}{\log \frac{L}{L_0}} = -f\left(\frac{\log \frac{X}{X_0}}{\log \frac{L}{L_0}}\right) \tag{49}$$

Here X_0 and L_0 can be thought of as physical units in which the quantities X and L are to be measured.

It is, at first glance, not easy to comprehend what this equation is telling us about how the the distribution of X looks!. Define a *local scaling index* α by the equation

$$\alpha = \frac{\log \frac{X}{X_0}}{\log \frac{L}{L_0}}. \tag{50}$$

Since α depends on the realization of the value of X in a cell of scale L, α is a possibly random, function of position. This is why it is referred to as a *local* scaling index. With this, the probability of finding a value X in a cell of size L is just

$$p(X, L) = p(X_0, L_0)\left(\frac{L}{L_0}\right)^{-f(\alpha)}. \tag{51}$$

We have power law scaling with cell size, but the scaling index α is an arbitrary function of the quantity X and the cell size L.

These two forms Eq. (48) and Eq. (49) of scaling agree only when $g(x)$ is a power law and $f(x)$ is linear.

If we look only at points such that α in Eq. (51) has some specific value, the distribution $p(X, L)$ has the form (48): the set of points with α = constant is a simple fractal of dimension $f(\alpha)$. Since the set consists of a range of values of α it can be called a *multifractal*, a set of intertwined simple fractals having different dimensions (see Fig. 16).

Note, however, this cautionary tale. A set of points distributed in power-law clusters is not necessarily a multifractal. It is only a multifractal if the scaling indices α are themselves

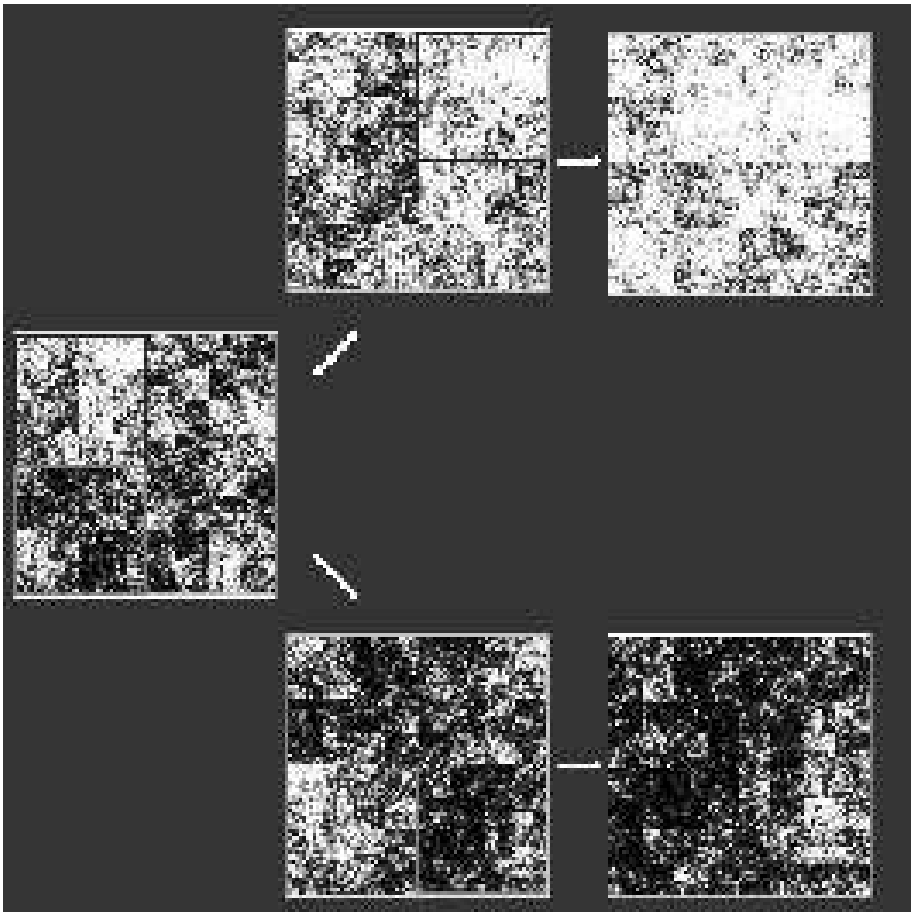

FIG. 16 A multifractal mass distribution over a square of side L_0. The distribution has been generated following a multiplicative cascade process (Martínez *et al.*, 1990). The gray scale represents the quantity of mass (X) in each pixel. Successive enlargements of two different regions of the original plot illustrate the inhomogeneity of the mass distribution.

constant on homogeneous fractal set. Thus not all point distributions are multifractals, even if they are distributed in power law clusters. A modified version of the scaling indices formalism, the "weighted scaling indices", has been recently introduced (Räth *et al.*, 2002). This method allows us to statistically quantify the local morphological properties of the galaxy distribution.

It can be shown that the descriptions of a point set via its statistical moments (38) or via the distribution of its scaling indices (51) are totally equivalent. The functions $f(\alpha)$ and $\tau(q)$ are related to one another via a Legendre transform (Jones *et al.*, 1992):

$$\tau(q) = \alpha q - f(\alpha), \qquad \alpha(q) = \frac{d\tau}{dq}.$$

VII. CLUSTERING MODELS

A. Cosmological simulations

1. Aarseth

The simplest way to explain the observed clustering is to do nonlinear numerical simulations of the galaxy clustering process. Although such simulations provide no deep explanations for what is going on, the ability to reproduce cosmic clustering simply by using a distribution of particles moving under their mutual gravitational interactions is quite striking.

N-body models have served to disprove several popular hypotheses on the evolution of large-scale structure, and motivated to introduce new assumptions. The downfall of the "Standard Cold Dark Matter Model" (SCDM) started with N-body models that gave top-heavy large-scale structure and too large pairwise velocity dispersion compared to the observations. Another example is the present controversy over cuspy centers of dark halos, which were found in high-resolution N-body simulations, but which are not observed. This motivated intensive study of Warm Dark Matter models.

The origin of N-body experiments as we know them today

is the work of Sverre Aarseth at Cambridge England (Aarseth, 1978). Aarseth was a student of Fred Hoyle whose visionary insight foresaw as long ago as 1965 the role that computers would play in astronomical research. Aarseth not only developed series of N-body codes tailor-made for different problems, he made these codes available to all and never even asked to be named as a collaborator.

The particle-particle codes developed by Aarseth were originally aimed at simulating problems in stellar dynamics. The particles were point masses and integrating of tight binaries was through two-body regularization. This was adapted to the cosmological problem by making the particles soft rather than point-like, and so dropping the need for the time-consuming calculation of binary encounters. The first papers using this modified code (Aarseth *et al.*, 1979; Gott *et al.*, 1979) used a mere 1000 equal mass particles and simple Poisson initial conditions. Yet they were able to reproduce a power-law correlation function for the clustering of these points.

2. Subsequent developments

During the 1970's the application of N-body codes to the problem of gravitational clustering mushroomed. Faster computers and improved numerical techniques drove particle numbers up. Following on from that work there has been a gradual growth in the number of particles used in simulations: 30,000 by the 1980s (Efstathiou *et al.*, 1985), 1,000,000 by Bertschinger and Gelb (1991) in the 1990s (see also the review Bertschinger (1998)) and Couchman *et al.* (1995), and now more than 1000,000,000 by the "Virgo Consortium" (Evrard *et al.*, 2002).

The N-body models cover a wide range of cosmic parameters and have enough particles to be used in trying to discriminate the clustering properties of the different models. We show in Fig. 17 a recent 10^9-point lightcone simulation of the "Virgo Consortium", a deep wedge $40h^{-1}$Mpc thick and $3.5h^{-1}$Gpc deep, extending to $z = 4.8$ (the universe was then about one eleventh of its present age). The upper sector of the "tie" shows a picture that we hope to get from the SDSS survey, a wider wedge reaching $z = 0.25$. Progressing in time from the largest redshift until present, we see how the structure gradually emerges. This simulation is described in Evrard *et al.* (2002).

3. Confronting with reality

Sometimes we might get the impression that N-Body simulations are better than the real thing, as in the game of 'Better Than Life' played by some of the characters in the BBC TV program *Red Dwarf*. In the early 1970's people were enthusiastic about a mere 1000 particles (which reproduced the correct two-point correlation function so "it had to be right"). They got even more enthusiastic with a million particles in the 1990's and now it is indeed better than life, especially with reality enhancing graphics, and ready-to-play in your PowerPoint presentation movies.

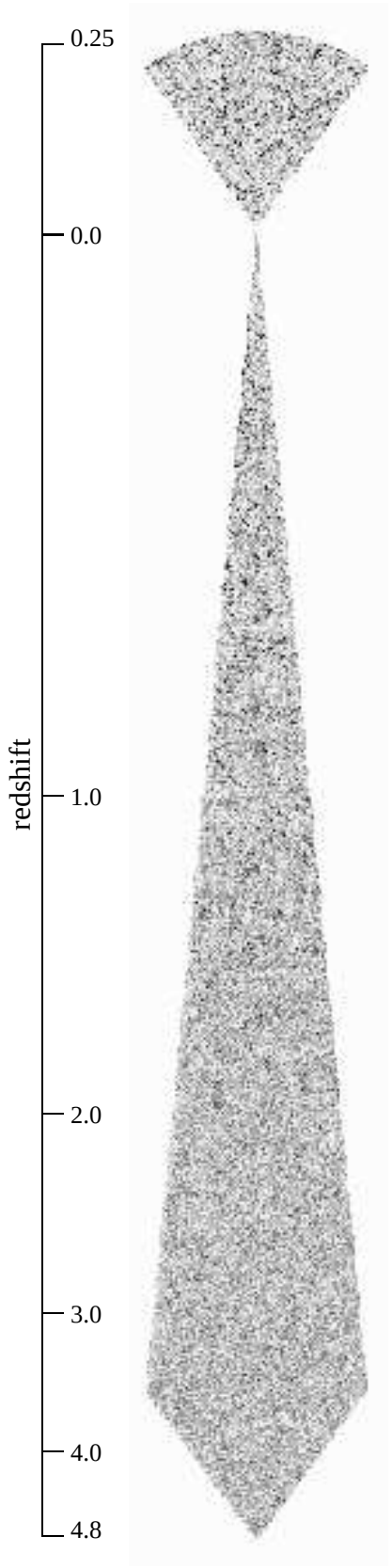

FIG. 17 A deep simulated wedge of the Universe. Figure by Gus Evrard and Andrzej Kudlicki, courtesy of the "Virgo Consortium"; details in text.

Is this enthusiasm justified? N-Body simulations are certainly a success story, and they certainly make a huge contribution to our understanding of cosmology. The models are nevertheless extremely limited simply because they lack any real gasdynamics, and star formation which must be important or other things that we know little about (such as magnetic fields, which one hopes are not important). There are some salutary lessons, such as the effects of discreteness in pure N-Body models (Splinter *et al.* (1998)), but there is little or no response to such points from the N-Body community at large. So maybe we should not worry and just bask in what is after all better than life.

Up until now, most comparisons between the results of numerical experiments and the data have been made simply in terms of the galaxy clustering correlation function. Even this is fraught with difficulty since the observed data concerns the distribution of light whereas the numerical models most readily yield the clustering properties of the gravitating matter, most of which may well be dark and invisible. The key ingredient that has to be added is star formation, and it is perhaps true to say that attempts at doing this have so far been simple heuristic first steps.

Another popular model result, the mass function (distribution of masses) of rich galaxy clusters, depends less on star formation problems, but knowledge of formation of galaxies and clusters is certainly necessary to compare the simulated and observed mass functions.

Some measures, such as the distribution of velocity dispersion of galaxies and the distribution of halo masses are independent of the mass-to-light problem, but it is only recently that the large scale redshift surveys and surveys of real gravitational lenses have begun to yield the kind of data that is required.

4. Scaling in dark matter halos

N-body simulations have revealed fascinating scaling problems of their own, mostly for smaller scales than those described in this review. As the initial power spectrum of perturbations is almost a power law for comoving scales less than $10h^{-1}$Mpc, and cold dark matter and gravitation do not bring in additional scales, the evolution of structure on these scales, and the final structure of objects should be similar.

As a proof of this conjecture, N-body simulations show that dark matter halos have well-defined universal density profiles. There is slight disagreement between the practitioners on the exact form of this profile, but the most popular density profile by far is that found by Navarro *et al.* (1996) (the NFW profile):

$$\rho(r)/\rho_c = \frac{\delta_c}{y(1+y)^2}, \quad y = r/r_s, \tag{52}$$

where ρ_c is the critical cosmological density, δ_c a characteristic density contrast, and r_s is a scale radius. The masses of N-body halos are usually defined as that contained within the "virial radius" r_{200}, the radius of a sphere of mean density contrast 200. Then the only parameter describing the NFW profile for a halo of given mass is the concentration ratio $c = r_{200}/r_s$.

There have been many studies with differing conclusions on the exact properties of dark halo profiles; we shall refer the reader to the latest accurate analysis (Navarro *et al.*, 2004). The main difficulty is in eliminating a multitude of possible numerical artifacts, but nobody seems to doubt that universal profiles exist. Concentration ratios depend on the mass of a halo, but this seems to be the main difference.

In connection with observations, the main problem has been the existence of a density cusp in the center of a halo, and the value for the logarithmic slope. As this demands probing the very central regions of galaxy clusters and galaxies, the problem is still open.

5. Scaling in galaxy properties

While the notion of the universal density profile arose from N-body simulations, other scaling laws for cluster- and galaxy-sized objects have observational origin. The best established law is called the Fundamental Plane (FP). This scaling law was discovered simultaneously by Djorgovski and Davis (1987) and Dressler *et al.* (1987). It is rather complex, meaning that elliptical and S0 (early-type) galaxies form a plane in the 3-space of $(\log L, \log r_c, \log \sigma)$, where L is the total luminosity of the galaxy, r_c is its characteristic radius and σ^2 its stellar velocity dispersion. (As L and r_c can be combined to give $\langle I \rangle_c$, the mean surface brightness of the galaxy, the latter is frequently chosen as one of the three variables.) These properties of elliptical galaxies are tightly correlated, and are thought to describe the process of their formation. Similar correlations have been discovered for galaxy clusters (Lanzoni *et al.*, 2004). Their existence demands special scaling for the mass-luminosity ratio of cluster galaxies with the mass of the cluster.

As the fundamental plane relation contains the size of a galaxy, it can be used for estimating the distance to a galaxy. Having a distance estimate, we can disentangle the proper velocity of a galaxy from that of the Hubble flow. Dressler *et al.* (1987) ("the Seven Samurai") used the newly discovered fundamental plane relation to derive for the first time the nearby large-scale galaxy velocity field. In this way the "Great Attractor", a large supercluster complex partly hidden by the Milky Way, was predicted by Lilje *et al.* (1986) from a relatively local sample of galaxies and discovered by Lynden-Bell *et al.* (1988) using a larger sample of elliptical galaxies. A recent example of a similar project is the NFP Survey (NOAO Fundamental Plane Survey), a survey of 100 rich X-ray selected clusters within $200h^{-1}$, where the fundamental plane of early-type cluster galaxies is used to determine cluster distances and, therefore, large scale cluster flows (Nelan *et al.* (2003)).

When talking about scaling laws at galaxy and cluster scales, one cannot bypass the well-known Tully-Fisher (Tully and Fisher, 1977) and Faber-Jackson (Faber and Jackson, 1976) scalings, which declare that the luminosities (or masses) of galaxies are tightly correlated

with their velocity spread. These scalings can be written as:

$$L \sim V_{\rm max}^{a}, \qquad \text{Tully-Fisher, spiral galaxies,}$$
$$L \sim \sigma^{a}, \qquad \text{Faber-Jackson, elliptical galaxies,}$$

where $V_{\rm max}$ is the maximum rotation velocity of a spiral galaxy, and σ^2 is the stellar velocity dispersion of an elliptical galaxy (in fact, the fundamental plane relation previously explained is a refinement of the Faber-Jackson relation). The power-law exponent $a \approx 4$, which can be easily explained, if there are no dark matter halos around galaxies, and is difficult to explain for the CDM paradigm. This difficulty has been of strong support for the MOND theory (Milgrom, 1983). This theory substitutes the Newtonian theory in the limit of small accelerations by an empirical formula, which explains the flat rotating curves of galaxies without invoking the notion of dark matter, and explains naturally the Tully-Fisher scaling. MOND does not fit into the present picture of fundamental physics, as the CDM assumption does, but it has found a number of followers. A critical (but well-meant) assessment of MOND can be found in a recent review by Binney (2003).

B. Statistical models

The earliest models of galaxy clustering were based on Charlier's simple notion that the system of galaxies formed some kind of simple hierarchy. There was little or no observational basis for such models. Later on, in the 1950's when galaxy clusters were seen as objects in their own right, the clustering process was seen as aggregates of points (the clusters) scattered randomly in an otherwise uniform background.

It was not until the systematic analysis of galaxy catalogs and the discovery of that the two-point clustering correlation function is a power law that the distribution of galaxies was seen as being a consequence of gravitational aggregation on all scales. Galaxy clustering was a general phenomenon and rich galaxy clusters were seen as something rather rare and special, but nevertheless a part of the overall clustering process.

1. Neyman-Scott processes

One of the most important of the early attempts to model the galaxy clustering process came from the Berkeley statisticians Neyman and Scott (1952). They sought to model the distribution of galaxy clusters as a random spatial superposition of groups of galaxies of varying size. The individual groups were to have their galaxies distributed in a Gaussian density distribution and they found evidence of superclusters (Neyman *et al.*, 1953).

Whereas the model in that early form had limited application for cosmology, the Neyman-Scott process became a discipline in its own right. It remains to be seen whether a generalization of these processes might be resurrected for present day clustering studies. A recent program in a similar vein is called the halo model; we shall describe it below.

2. Simple fractal models

There has for a long time been a strong interest in the theory of random processes which has had a strong impact on many fields of physics (see for example the collection of classic papers by Wax (1954)). Among the simplest of random processes is the so-called "Random Walk" in which a particle continually moves a random distance in a random direction subject to a set of simple rules. The collection of points at which the particle stops before moving on has a distribution that can often be calculated.

Many random walks result in distributions of points that are clustered. The character of the clustering depends on the conditions of the walk. It did not take long before someone suggested that the galaxy distribution could be modeled by a random walk (Fournier d'Albe, 1907; Mandelbrot, 1975).

What was of interest in these random walk models is that they could be characterized by a single parameter: a power law index that related to the mean density profile of the point distribution.

It should be noted that these simple fractal models have little direct interest in cosmology: they are merely particularly simple examples of clustering processes among many. In particular they do not show the transition to cosmic homogeneity on large scales and have no relevant dynamical content.

That is not to say that one cannot construct relevant fractal models. By 'relevant' we mean that the model should at least be consistent with or derived from some dynamical theory for the clustering: anything else is merely descriptive. Some relevant ones are described below.

3. More complex clustering models

It was clear at an early stage that the two-point correlation function for galaxy clustering was by itself an incomplete descriptor of the galaxy distribution: quite different point distributions can have the same two-point correlation function.

The obvious step was to compute 3-point and higher order correlation functions and to seek a more complete description of the clustering that way. The key discovery was that the higher order functions could all be expressed as sums of products of two-point correlation functions alone. This lead to a quest to build clustering hierarchies that embodied these important scaling properties.

It was evident at the outset that such models would have to be more sophisticated than the simple fractal hierarchy of Mandelbrot. The first such model was the clustering hierarchy (a bounded fractal) of Soneira and Peebles (1978) . This model produced a galaxy distribution looking remarkably like the observed galaxy distribution.

The observation that the galaxy distribution was a clustering hierarchy in which all orders of correlation function could be related to the basic two-point function could be described in another way. Instead of using just one power law index, as in a simple fractal, to describe the clustering process, it might be possible to use a distribution of power laws. This gave rise to the multifractal model of Jones *et al.* (1988) in which

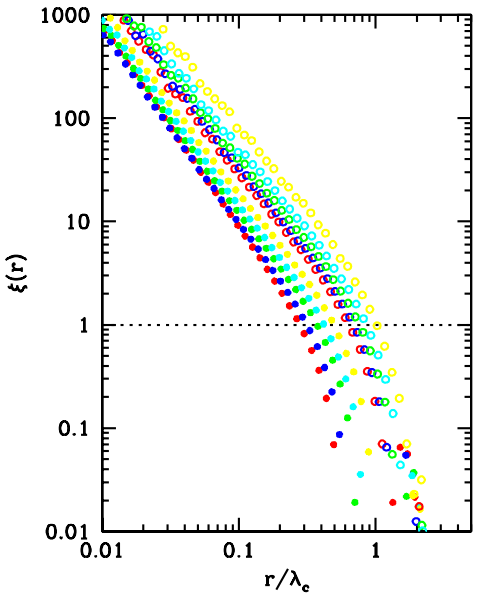

FIG. 18 The scaling of the two-point correlation function is shown for different subsamples of a Voronoi vertices model. The subsamples have been selected according to given "richness" criterium that mimic that of the real galaxy cluters, from van de Weygaert (2003).

the distribution could be generated by a set of simple scaling laws.

4. Voronoi tesselations

The Voronoi tessellation, and the related Delaunay tessellation, provide well-known tools for investigations into clustering in point processes. The construction of such a tessellation starts from a set of seed points distributed randomly according to some rule (often Poisson distributed). A set of walls is constructed around each point defining a closed polyhedron. Every point in the polyhedron has the original seed point as its nearest point among the set of all seeds.

The polyhedron effectively defines a volume of influence for each seed point. The vertices of these polyhedra define a set of points that is also randomly distributed, but in a way that is quite different from the distribution of the original seeds.

These tessellations were introduced into astronomy by Icke and van de Weygaert (1987) as a model for the galaxy clustering process. Detailed description of two- dimensional Voronoi tessellations can be found in Ripley (1981). The best sources of information on 3-dimensional tessellations in general and in cosmology are van de Weygaert (1991, 2002).

What is remarkable is that the two point correlation function for the Voronoi Vertices generated from Poisson distributed seeds is a power law that is close to the observed power law of the two-point galaxy correlation function (see Fig. 18). This tessellation thereby provides a possible model for the observed galaxy distribution.

Galaxies appear to form on filaments and sheets that surround void regions. If in the Voronoi model we regard the original seeds as the centers of expansion of cosmic voids, this model becomes a dynamically plausible nonlinear model for the formation large-scale structure formation (van de Weygaert and Icke, 1989). The resulting galaxy distribution has many interesting features that seem to accord with the distribution of galaxies in redshift surveys (Goldwirth *et al.*, 1995).

5. Lognormal models and the like

A rather simplistic yet effective model was presented by Coles and Jones (1991) who postulated that the originally Gaussian density field would evolve into a log-normally distributed density field. The motivation for this was simply that the hydrodynamic continuity equation implied that $\log \rho$ would be normally distributed if the velocity field remained Gaussian. The counts in cells of various size for N-body models and for catalogs of galaxies are indeed approximately log-normal for a variety of cell volumes.

Clearly, the contours where the density equaled the mean would remain fixed: there is no dynamics in such a model. Such a naive approach could never reproduce the structure we see today.

There are several relatively simple generalizations of the lognormal distribution, notably the Poisson lognormal (Borgani, 1993) and the negative binomial distribution (Betancort-Rijo, 2000; Elizalde and Gaztañaga, 1992).

6. Saslaw-Sheth models

A novel set of distribution functions was introduced by Saslaw and Sheth (1993) and Sheth and Saslaw (1996) derived from a thermodynamic description of the clustering process. The distribution functions describe the probability that a randomly chosen sample volume contains precisely N galaxies. There is only one free parameter in terms of which the count distribution for arbitrary values of the volume can be fitted. The resulting fit is quite remarkable for both N-body experiments and for the data sets that have been analysed (Saslaw and Crane, 1991).

The distribution function has some interesting scaling properties that are discussed in Saslaw (2000).

Given the quality of the fit to the data, this is clearly a model in which the underlying physical motivation deserves more attention.

7. Balian and Shaeffer

An alternative approach is to create a model for the evolution some statistically important quantities. Balian and Schaeffer (1989a) selected the Void Probability Function: the probability that a volume V chosen at random would contain no points (galaxies). This can be generalized to discuss the probability distributions of volumes containing 1, 2 or N galaxies.

Balian and Shaeffer were able to express many of the details of the clustering hierarchy in terms of the Void Probability function, in particular they found a bifractal behavior for the galaxy distribution (Balian and Schaeffer, 1989b). Scaling of voids as a test of fractality has been studied by Gaite and Manrubia (2002).

The mass (luminosity) function was also derived from similar scaling arguments by Bernardeau and Schaeffer (1991), who found the scaling between the galaxy and cluster luminosity functions to support the theory of Balian and Schaeffer (1989b).

Vergassola *et al.* (1994) attacked the problem of gravitational evolution of hierarchical (fractal) initial conditions. They choose the adhesion approximation to describe the gravitational dynamics and demonstrated (with much greater rigour than usual in cosmological papers) that the mass function has two scaling regimes, defined by the scaling exponent of the initial velocity field. This is the only paper that explicitly describes the evolution of structure on all, even infinitesimally small scales.

C. Dynamical models

1. Stable clustering models

The earliest attempt to explain the apparent power law nature of the two point galaxy correlation function was due to Peebles (1974a,b) and to Gott and Rees (1975). These models were based on the simple idea that a succession of scales would collapse out of the expanding background and then settle into some kind of virial equilibrium. The input data for the model was a power law spectrum of primordial imhomogeneities and the output was a power law correlation function on those scales that had achieved virial equilibrium. There would, according to this model, be another power law on larger scales that had not yet achieved virial equilibrium.

For a primordial spectrum of the form $\mathcal{P}(k) \propto k^{-n}$ the slope of the two-point galaxy correlation function would be $\gamma = (3n + 9)/(n + 5)$, which for $n = 0$ gave a respectable $\gamma = 1.8$, while $n = 1$ gave an almost respectable $\gamma = 2$.

The apparent success of such an elementary model gave great impetus to the field: we saw something we had some hope of understanding. However, there were several fundamental flaws in the underlying assumptions, not the least of which was that the observed clustering power law extended to such large scales that virial equilibrium was out of the question. There were also complications arising out of the use of spherical collapse models for calculating densities.

Addressing these problems gave rise to a plethora of papers on this subject, too numerous to detail here. A fine modern attempt at this is Sheth and Tormen (1999). The subject has since evolved into some of the more sophisticated models for the evolution of large scale structure that are discussed later (e.g. Sheth and van de Weygaert (2004)).

2. BBGKY hierarchy

Cosmic structure grows by the action of gravitational forces on finite amplitude initial density fluctuations with a given power spectrum. We see these fluctuations in the COBE anisotropy maps and we believe they are Gaussian. This means that the initial conditions are described as a random process with a given two-point correlation function. There are no other higher order correlations: these must grow as a consequence of dynamical processes.

Given that, it is natural to try to model the initial growth of the clustering via a BBGKY hierarchy of equations which describe the growth of the higher order correlation functions. The first attempt in this direction was made by Fall and Severne (1976) though the paper by Davis and Peebles (1977) has certainly been more influential. The full BBGKY theory of structure formation in cosmology is described in Peebles (1980) and in a series of papers by Fry (Fry, 1982, 1984a). Fry (1985) predicted the 1-point density distribution function in the BBGKY theory. He also developed the perturbation theory of structure formation (Fry, 1984b), which has become popular again (see the recent review by Bernardeau *et al.* (2002)).

In the perturbative approach, the main question is how many orders of perturbation theory are required to give sensible results in the nonlinear regime.

3. Pancake and adhesion models

Very early in the study of clustering, Zel'dovich (1970) presented a remarkably simple, yet effective, model for the evolution of galaxy clustering. In that model, the gravitational potential in which the galaxies moved was considered to be known at all times in terms of the initial conditions. The particles (galaxies) then moved kinematically in this field without modifying it. They were in effect test particles with no self-gravity. The equations of motion were arranged so as to give the correct initial, small amplitude, linear approximation result.

The Zel'dovich model provided a first glimpse of the possible growth of large scale cosmic structure and led to the prediction that the galaxy distribution would consist of narrow filaments of galaxies surrounding large voids. Nothing of the sort had been observed at the time, but striking confirmation was later achieved by the CfA-II Slice sample of de Lapparent *et al.* (1986) whose redshift survey revealed for the first time remarkable structures of the kind predicted by Zel'dovich.

In order to make further progress it was necessary to cure one problem with the Zel'dovich model: the filaments formed at one specific instance and then dissolved. The dissolution of the filaments happened because there was nothing to bind the particles to the filaments: after the particles entered a filament, they left. The cure was simple: make the particles sticky. This gave rise to a new series of models, referred to a "adhesion models" (Gurbatov *et al.*, 1989a; Kofman *et al.*, 1992). They were based around the three dimensional Burgers equation.

In these models structure formed and once it formed it stayed put: the lack of self gravity within these models prevented taking them any further.

It was, however, possible to compute the scaling indices for various physical quantities in the adhesion model. This was achieved by Jones (1999) using path integrals to solve the relevant version of the Burgers equation.

4. Renormalization group

Peebles (1985) first recognized that power law clustering might be described by a renormalization group approach in which each part of the Universe behaves as a rescaled version of the large part of the Universe in which it is embedded. This allows for a recursive method of generating cosmic structure, the outcome of which is a power law correlation function that is consistent with the dynamics of the clustering process.

Peebles (1985) used this approach for numerical simulations of the evolution of structure, hoping that the renormalization approach would complement the usual N-body methods, improving the usually insufficient spatial resolution and helping to get rid of the transients caused by imperfect initial conditions. The first numerical renormalization model had only 1000 particles and suffered from serious shot noise.

This was later repeated on a much larger scale by Couchman and Peebles (1998). As before, they found that the renormalization solution produces a stable correlation function. However, the spatial structures generated by the renormalization algorithm differed from those obtained by the conventional test simulation. The relative velocity dispersion was smaller, and the mass distribution of groups was different. As a rule, the renormalization solution described small scales better, and the conventional solution was a better description of the large-scale structure. As both approaches, the conventional and the renormalization procedures, suffer from numerical difficulties, the question of a true simulation remains open.

5. The halo model and PThalo model

The early statistical model (Neyman and Scott, 1952) for the galaxy distribution assumed Poissonian distribution of clusters of galaxies. This model was resurrected by Scherrer and Bertschinger (1991) and has found wide popularity in recent years (see the review by Cooray and Sheth (2002)). In its present incarnation, the halo model describes nonlinear structures as virialized dark matter halos of different mass, placing them in space according to the linear large-scale density field that is completely described by the initial power spectrum. Such substitution is shown in Fig. 19, where the exact nonlinear model matter distribution is compared with its halo model representation.

Once the model for dark matter distribution has been created, the halos can be populated by galaxies, following different recipes. This approach has been surprisingly fruitful, allowing calculation of the correlation functions and power spectra, prediction of gravitational lensing effects, etc. This also tells us that low-order (or any-order) correlations cannot be the final truth, as the two panels in Fig. 19 are manifestly different.

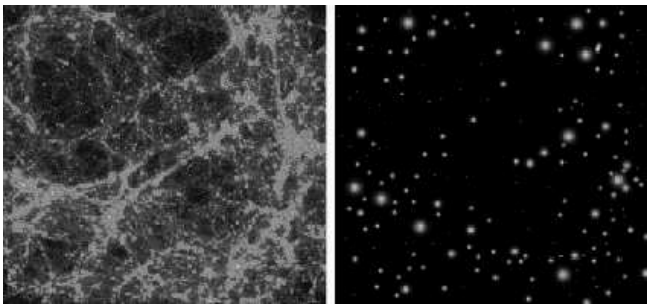

FIG. 19 The halo model. The simulated dark matter distribution (left panel) and its halo model (right panel), from Cooray and Sheth (2002).

The success of the (statistical) halo model motivated a new dynamical model to describe the evolution of structure (Scoccimarro and Sheth, 2002). The PTHalos formalism, as it is called, creates the large-scale structure using a second order Lagrangian perturbation theory (PT) to derive the positions and velocities of particles, and collects then particles into virialized halos, just as in the halo model. As this approach is much faster than the conventional N-body simulations, it can be used to sample large parameter spaces – a necessary requirement for application of maximum likelihood methods.

6. More advanced models

Two analytic models in the spirit of the Press-Schechter density patch model are particuarly noteworthy: the "Peak Patch" model of Bond and Myers (Bond and Myers, 1996a,b,c) and the very recent "Void Hierarchy" model of Sheth and van de Weygaert (2004).

Both of these models attempt to model the evolution of structure by breaking down the structure into elements whose individual evolution is understood in terms of a relatively simple model. The overall picture is then synthesized by combining these elements according to some recipe. This last synthesis step is in both cases highly complex, but it is this last step that extends these works far beyond other like-minded approaches and that lends these models their high degree of credibility.

The Peak Patch approach is to look at density enhancements, while the Void Hierarchy approach focusses on the density deficits that are likely to become voids or are embedded in regions that will become overdensities. It somewhat surprising that Peak Patch did not stimulate further work since, despite its complexity, it is obviously a good way to go if one wishes to understand the evolution of denser structures.

The Void Hierarchy approach seems to be particularly strong when it comes to explaining how large scale structure has evolved: it views the evolution of large scale structure as being dominated by a complex hierarchy of voids expanding

to push matter around and so organize it into the observe large scale structures. At any cosmic epoch the voids have a size distribution which is well-peaked about a characteristic void size which evolves self-similarly in time.

D. Hydrodynamic models for clustering

Let the physical position of a particle at some (Newtonian) time t be $\mathbf{r}$. It is useful to rescale this by the background scale factor $a(t)$ and label the particle with its comoving coordinate

$$\mathbf{x} = \frac{1}{a(t)}\mathbf{r} \tag{53}$$

relative to the uniform background. Formation of structure means that viewed from a frame that is co-expanding with the background, particles are moving and the values of their coordinates $\mathbf{x}$ are changing in time.

There is another coordinate system that can be used: the Lagrangian coordinate $\mathbf{q}$ of each particle. $\mathbf{q}$ can be taken to be the value of the comoving coordinate $\mathbf{x}$ at some fiducial time, usually at $t = 0$ (the Big Bang) or a little later, and so remains fixed for each particle. The transformation between the Lagrangian coordinate $\mathbf{q}$ and the proper (Eulerian) coordinate $\mathbf{x}$ is achieved via the equations of motion (see for example Buchert (1992)).

In a homogeneous universe, the particle velocity in physical coordinates is $\dot{\mathbf{r}} = H\mathbf{r}$, where $H = \dot{a}/a$ is the Hubble expansion rate. In this situation the comoving coordinate $\mathbf{x}$ of a particle is fixed and there is no peculiar velocity relative to the co-expanding background coordinate system.

In an inhomogeneous universe, the displacement of the particles relative to the co-expanding background coordinate system, $\mathbf{x}$ is time dependent. The velocity relative to these coordinates is just $\dot{\mathbf{x}}$, and this translates back to a physical "peculiar" velocity $\mathbf{v} = a\dot{\mathbf{x}}$. We can therefore write the total physical velocity of the particle (including the cosmic expansion) as

$$\mathbf{V} = \mathbf{v} + H\mathbf{r}, \qquad \mathbf{v} = a\dot{\mathbf{x}},$$

where here the dot derivative is the simple time derivative taken at a fixed place in the co-expanding frame.

1. Cosmological gas dynamics

As usual, we work in the standard comoving coordinates $\{\mathbf{x}\}$ defined by rescaling the physical coordinates $\{\mathbf{r}\}$ by the cosmic scale factor $a(t)$, as described above.

The motion of a particle is governed by the equations of momentum conservation, the continuity equation and the Poisson equation. Expressed relative to the comoving coordinate frame and in terms of density fluctuation δ relative to the mean density $\rho_0(t)$:

$$\delta(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t) - \rho_0(t)}{\rho_0(t)}, \tag{54}$$

these equations are (Munshi and Starobinsky (1994); Peebles (1980)):

$$\frac{\partial}{\partial t}(a\mathbf{v}) + (\mathbf{v}.\nabla)\mathbf{v} = -\frac{\partial\phi}{\partial\mathbf{x}}, \qquad \text{momentum conservation,} \tag{55}$$

$$\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla.[(1+\delta)\mathbf{v}] = 0, \qquad \text{continuity,} \tag{56}$$

$$\frac{\partial^2\phi}{\partial\mathbf{x^2}} = 4\pi G\rho_0 a^2\delta(\mathbf{x}, t), \qquad \text{Poisson.} \tag{57}$$

Here $\phi(\mathbf{x}, t)$ is the part of the gravitational potential field induced by the fluctuating part of the matter density $\rho(\mathbf{x}, t)$ relative to the mean cosmic density $\bar{\rho}(t)$. G is the Newtonian gravitational constant.

Note that here the source of the gravitational potential is the same density fluctuations that drive the motion of the material with velocity $\mathbf{u}(\mathbf{x})$.

2. The cosmic Bernoulli equation

It can be assumed throughout that the cosmic flow is initially irrotational; this is justified by the fact that rotational modes decay during the initial growth of structure or from CMB data. This assumption makes it possible to take the next step of introducing a velocity potential that completely describes the fluid flow and then going on to get the first integral of the momentum equation: the Bernoulli equation.

Introduce a velocity potential $\mathcal{V}$ such that

$$\mathbf{v} = -\nabla\mathcal{V}/a, \tag{58}$$

Recalling that the gradient operator is taken with respect to the comoving $\mathbf{x}$ coordinates, we see that $\mathcal{V}$ is the usual velocity potential for the real flow field $\mathbf{v}$. The first integral of the momentum equation becomes

$$\frac{\partial\mathcal{V}}{\partial t} - \frac{1}{2a^2}(\nabla\mathcal{V})^2 = \phi, \tag{59}$$

This is referred to as the Bernoulli equation, though in fluid mechanics we usually find an additional term: the enthalpy w defined by $\nabla w = (\nabla p)/\rho$. This vanishes in the post-recombination cosmological context by virtue of neglecting pressure gradients.

As a matter of interest, for a general (non-potential) flow we have an integral of the momentum equation that is a constant only along flow streamlines. Different streamlines can have different values for this constant. It is only in the case of potential flow such as is supposed here that the constant must be the same on all streamlines.

The Bernoulli equation (59) is a simple expression of the way in which the velocity potential (described by $\mathcal{V}$) is driven by a gravitational potential ϕ in a uniform expanding background (described by the expansion scale factor $a(t)$). Despite its simplicity it has several drawbacks, the most serious

of which is the fact that an additional equation (the Poisson equation in the form (57)) or simplifying assumption is needed to determine the spatially fluctuating gravitational potential $\phi(\mathbf{x})$.

Another drawback of the Bernoulli equation as presented here is that it describes a dissipationless flow: there is no viscosity. Dissipation, be it viscosity or thermal energy transfer, is an essential ingredient of any theory of galaxy formation since there has to be a mechanism for allowing the growth of extreme density contrasts. Galaxy formation is not an adiabatic process!

A difficulty that presents itself with Eq. (59) is that the term involving the spatial derivative of the velocity potential, $\nabla\mathcal{V}$ is multiplied by a function of time $a(t)$. This can be removed by a further transformation of the velocity potential:

$$\mathcal{U} = \frac{\mathcal{V}}{a^2\dot{a}} \tag{60}$$

Now, the potential $\mathcal{U}$ is related to the comoving peculiar velocity field $\mathbf{u}$ by $\mathbf{u} = -a\dot{a}\nabla\mathcal{U}$. In terms of this rescaled potential the Bernoulli equation takes on a form that is more familiar in hydrodynamics:

$$\frac{\partial\mathcal{U}}{\partial a} - \frac{1}{2}(\nabla\mathcal{U})^2 = \frac{3}{2a}(A\phi - \mathcal{U}). \tag{61}$$

Here we have used the scale factor $a \propto t^{2/3}$ as the time variable, and noted that $A = -(3\dot{a}a^2)^{-1} =$ constant in an Einstein–de Sitter Universe (Kofman and Shandarin, 1990, 1988) (NB.: in these papers the velocity potential has the opposite sign from ours).

3. Zel'dovich approximation

The Zel'dovich approximation (Shandarin and Zel'dovich (1989); Zel'dovich (1970)) to the cosmic fluid flow was a remarkable first try at describing the appearance of the large scale structure of the Universe in terms of structures referred to as "pancakes" and "filaments" that surround "voids". Indeed, one might say that through this approximation Zel'dovich predicted the existence of the structures mapped later by de Lapparent *et al.* (1986).

The Zel'dovich approximation is recovered from the last variant of the Bernoulli equation above (61) by setting $A\phi = -\mathcal{U}$. This latter relationship replaces the Poisson equation in that approximation.

While predicting the qualitative features of large scale structure, the Zel'dovich approximation had a number of shortcomings, notable among which was the fact that particles passed through the pancakes rather than getting stopped there and accumulating into substructures (galaxies and groups).

The last decade has seen a host of improvements to the basic prescription which are nicely reviewed by Buchert (1996); Susperregi and Buchert (1997) and by Sahni and Coles (1995). These improvements largely fall into three categories: "adhesion" schemes in which particle orbits are prevented from crossing by introducing an artificial viscosity, various "fixup" schemes in which simplifying assumptions are made about the gravitational potential or the power spectrum and "nonlinear" schemes in which the basic Zel'dovich approximation is taken to a higher order. We defer the discussion of the "adhesion approach" to the next section.

4. Super-Zel'dovich approximations

Several recipes have been given for improving on the Zel'dovich approximation in its original nondissipative form without introducing an *ad hoc* artificial viscosity. In these approximations, the Poisson equation is replaced with some *ansatz* regarding the gravitational potential: it can be set, for example, equal to a constant, or equal to the velocity potential.

Matarrese *et al.* (1992); Melott *et al.* (1994a) introduced a variant called the "Frozen Flow Approximation" (FFA) in which the peculiar velocity field at any point fixed in the background is frozen at its original value: the flow is "steady" in the comoving frame. (The initial peculiar velocity field is chosen self-consistently with the fluctuating potential and the initial density field).

In another approach Bagla and Padmanabhan (1994, 1995) and Brainerd *et al.* (1993) assume that the fluctuating part of the gravitational potential at a point expanding with the background remains constant (as it does in linear theory). This is referred to as the "Frozen Potential Approximation" (FPA) or "Linearly Evolving Potential" (LEP). The motivation for this as a nonlinear extension arises from some special cases where nonlinear calculations have been done and from N-body simulations in which the potential is seen not to change much in comparison with other quantities.

Munshi and Starobinsky (1994) point out that the standard Zel'dovich approximation is equivalent to the assumption that $\mathcal{V} = \phi t$, while the Frozen Flow approximation is $\mathcal{V} = \phi_0 t$ and the Frozen (or Linear) Potential approximation is $\Phi = \phi_0$. In any case, this last equation provides an equation for the velocity potential given a model for the gravitational potential.

More recently, we have seen the "Truncated" Zel'dovich Approximation (Coles *et al.*, 1993; Melott *et al.*, 1994b), the "Optimized" Zel'dovich Approximation (Melott *et al.*, 1994c) and the "Completed" Zel'dovich Approximation (Betancort-Rijo and López-Corredoira, 2000) . These correlate remarkably well with full N-body simulations.

5. Nonlinear enhancements

Various authors have presented nonlinear versions of the Zel'dovich approximation. Gramann (1993); Susperregi and Buchert (1997) used a second order extension, while Buchert (1994) presented a perturbation scheme that is correct to third order in small quantities.

E. Nonlinear dynamic models

The Zel'dovich approximation and its fixups are Lagrangian descriptions of the cosmic fluid flow. Their impor-

tance lies in the fact that they capture the gross elements of the nonlinear clustering while their weakness lies in their sidestepping any real gravitational forces. Consequently, they have been used mainly as short-cut simulators of the evolution of large scale structure. Little analytic work has been done using these approaches.

It is the Lagrangian nature of those equations that makes it difficult to perform analytic calculations that might lead to an understanding of how, say, the two-point correlation function evolves with time. In order to make progress on an analytic front it is necessary to cast the equations into analytically tractable Eulerian forms. The basis for this was provided by the important "adhesion approximation" of Gurbatov *et al.* (1989b), though in the form presented there it was only ever used for numerical simulations.

1. Adhesion Approximations

The paper by Gurbatov *et al.* (1989b) provided a version of the Zel'dovich approximation in which particle shell-crossing was inhibited: the material was stopped as it approached the pancakes by an artificial viscosity introduced on a fairly ad hoc basis into the equations. The underlying equation in this approximation turns out to be the three dimensional Burgers Equation, and so the approach has the virtues of being simple to use and very easy to compute (see for example Weinberg and Gunn (1990)).

The adhesion approximation is in a sense a linear approximation: it is allowed to evolve into the nonlinear regime in the expectation that its behavior will mimic the nonlinear behavior. This shortcoming has recently been tackled by Menci (2002).

Just as the simple Zel'dovich approximation tends to diffuse the pancakes, the adhesion approximation ensures that asymptotically they are infinitely thin, and that the particle velocity perpendicular to these surfaces is zero. The slowing down of the particles as they approach the pancakes, the notion of "viscosity" in dark matter, and the lack of a full treatment of the gravitational field fluctuations leaves open some questions as to just how good the approximation is for studying, say, large scale cosmic flow fields.

It is remarkable how much can be done within the framework of the adhesion model. Babul and Starkman (1992) had introduced structure functions based on the moments of inertia of the local particle distribution, to describe the local shape of the matter distribution. They showed this to be a useful descriptor of the topology of the galaxy distribution. The evolution of these structure functions was studied analytically by Sathyaprakash *et al.* (1996). They analyzed the emergence of large scale filamentary and pancake-like structures and showed how this might lead to a large scale coherence in the galaxy distribution. Sahni *et al.* (1994) discussed the evolution of voids using the adhesion approximation. In their model, ever larger voids emerge at successive epochs, eventually leaving the largest voids. According to this model, voids contain some internal filamentary and pancake-like substructures that dissolve as the voids get older.

2. The Random Heat Equation

The random heat equation was introduced into the subject of cosmic structure evolution by Jones (1999). The Bernoulli equation (59), modified by introducing viscosity (see Jones (1999)), can be linearised by means of the Hopf-Cole transformation of variables in which we replace the velocity potential $\mathcal{V}$ with a logarithmic velocity potential ψ:

$$\mathcal{V} = -2\nu \ln \psi \tag{62}$$

If the gravitational potential is rescaled with the viscosity:

$$\phi(\mathbf{x}) = 2\nu\epsilon(\mathbf{x}), \tag{63}$$

Equation (59) with the viscosity term reduces to

$$\frac{\partial\psi}{\partial t} = \frac{1}{a^2}\nu\frac{\partial^2\psi}{\partial \mathbf{x^2}} + \epsilon(\mathbf{x})\psi. \tag{64}$$

Again, it is worth stressing that ν can depend on time, but we see that invoking a time dependence in ν means that the new potential $\epsilon(\mathbf{x})$ gains an explicit time dependence.

This time dependence can be masked so as to give the random heat equation in its standard form:

$$\frac{\partial\psi}{\partial t} = \nu\frac{\partial^2\psi}{\partial \mathbf{x^2}} + \epsilon(\mathbf{x})\psi. \tag{65}$$

It is now to be understood that either ν or ϵ (or both) may contain an explicit time dependence through a multiplying factor.

The renormalised potential field $\epsilon(\mathbf{x})$ is considered as given and the task is to find the potential ψ. This equation is familiar in slightly different forms in a variety of fields of physics where it has a variety of names: the Anderson Model, the Landau-Ginzburg equation, and with a complex time it is simply the Schrodinger Equation of quantum mechanics. We may hope to benefit from the vast knowledge that already exists about this equation.

If we take the limit $\nu \to 0$ and use the definition $\mathcal{V} = -2\nu \ln \psi$, we are led straight back to the familiar looking dynamical equation

$$\frac{\partial(a\mathbf{v})}{\partial t} = \nabla\phi,$$

telling us that the gravitational potential drives the fluctuating velocity field. Despite the circuitous route used in deriving the random heat equation, it still remains very close to the fundamental physical process that drives the growth of the large scale structure.

3. The Solution of the RH equation

We can formally solve random heat equation following the discussion of Brax (1992) (but see also Zel'dovich *et al.* (1985, 1987)). The solution is expressed in terms of path integrals as was first given by Feynman and Kac:

$$\psi(\mathbf{x}, t) = \int K(\mathbf{x}, t, \mathbf{x}_0, 0)\psi(\mathbf{x_0}, 0)d\mathbf{x_0},$$

where the propagator K is

$$K(\mathbf{x}_1, t, \mathbf{x}_0, 0) = \int_{\mathbf{x}(0)=\mathbf{x}_0}^{\mathbf{x}(t)=\mathbf{x}_1} e^{S(\mathbf{x}(\tau),\tau)} \mathcal{D}[\mathbf{x}(\tau)] \qquad (66)$$

and

$$S(\mathbf{x}(\tau), \tau) = -\int_0^\tau \left[\frac{1}{4\nu} \left| \frac{d\mathbf{x}(\tau')}{d\tau'} \right|^2 - \epsilon(\mathbf{x}(\tau'), \tau') \right] d\tau'$$

is the action. This is just the "free particle" action with an additional contribution to the action from the potential $\epsilon(\mathbf{x}, t)$ evaluated at appropriate places along the various paths that contribute to the solution (Brax (1992)). The integrand is just the Lagrangian for a particle moving in a potential $\epsilon(\mathbf{x}, t)$.

What is important here is that the potential $\epsilon(\mathbf{x}, t)$ contributes to the sum over all paths through an exponential. Thus the additive contributions from each part of the relevant paths results in a multiplicative contribution to the final solution. It is this which creates the lognormal distribution in $\psi(\mathbf{x}, t)$ if the potential $\epsilon(\mathbf{x}, t)$ in normally distributed.

4. Statistical Moments

Zel'dovich *et al.* (1985, 1987) explain the solution $\psi(\mathbf{x}, t)$ in straightforward terms. They point out that, of all the paths that contribute to the integral, one might expect the dominant contribution to come from those paths that pass rapidly through high maxima of this potential. However, there are rarer paths (optimal trajectories) that are traversed more quickly and so probe a greater volume that can encounter still larger (and rarer) maxima of the potential. These latter paths in fact make the main contribution to the integral. This is presented rigorously by Gärtner and Molchanov (1992).

The outcome of the discussion is that the moments of the distribution of ψ scale as

$$\langle \psi^q \rangle \propto \exp((q\bar{\epsilon} + \frac{1}{2} q^2 \sigma^2) t) \qquad (67)$$

where $\bar{\epsilon}$ and σ are the mean and variance of the process ϵ. This gives intermittency indices

$$\mu_q \propto (q^2 - q) \qquad (68)$$

(Brax, 1992), where the constant of proportionality is determined by the dimensional characteristics of the random process $\epsilon(\mathbf{x})$. Thus the solution of the random heat equation is lognormally distributed for a Gaussian fluctuating gravitational potential.

In view of the Hopf-Cole transformation, the velocity potential is in fact the logarithm of the pseudo-potential ψ: $\mathcal{V} = -2\nu \ln \psi$. Since ψ is lognormally distributed, it follows that $\mathcal{V}$ is normally distributed and we can compute its rms error as

$$\sigma_{\mathcal{V}} \propto \sigma_\phi t^{\frac{1}{2}} \qquad (69)$$

Remember that the variance of the gravitational potential fluctuations σ_ϕ^2 may itself have a time dependence. This is one of the things that was assumed as given and which in the single-component model is given by the approximation used to eliminate the Poisson equation.

5. The Schrodinger Equation

Starting with the coupled Klein-Gordon and Einstein field equations, Widrow and Kaiser (1993) produced an ansatz for replacing the Euler and continuity equations of hydrodynamics with a Schrodinger equation in the form

$$i\mathcal{H}\frac{\partial \Psi}{\partial t} = -\frac{\mathcal{H}^2}{2m}\frac{\partial^2 \Psi}{\partial \mathbf{x^2}} + m\phi(\mathbf{x})\Psi. \qquad (70)$$

(see also Speigel (1980)). $\mathcal{H}$ here is taken to be an adjustable parameter controlling spatial resolution. In this model the gravitational potential and density fields are given by

$$\frac{\partial^2 \phi}{\partial \mathbf{x^2}} = 4\pi G \Psi \Psi^*, \qquad \rho = |\Psi^2| \qquad (71)$$

Widrow and Kaiser (1993) see this as a means for doing numerical simulations of the evolution of large scale structure (they use a Schrodinger solver based on an implicit finite differencing method called Cayley's Scheme).

The Schrodinger equation for Ψ can be solved analytically by identical procedures to those described above for solving the random heat equation, the difference being that the potential Ψ being solved is complex. Ψ is directly related to the density field. This route is advocated by Coles (2002) in his very clear discussion of models for the origin of spatial intermittency. Coles and Spencer (2003) have taken this further and shown how to add effects of gas pressure corresponding to a polytropic equation of state. They present this as a useful approach for modeling the growth of fluctuations in the mildly nonlinear regime, which is somewhat short of the ambition of the original Jones (1999) program.

6. General Comments

The relative merits of the random heat equation and the Schrodinger equation approach are yet to be assessed. They are derived from quite different premises: one pretends to be a derivation from the basic equations while the other is an ansatz based on interpreting quantum mechanics as a fluid process. Each has a level of arbitrariness: one involves an unknown (unphysical) viscosity that is allowed to tend to zero, while the other involves a tuning parameter, the effective Planck Constant $\mathcal{H}$ that can probably be allowed to become vanishingly small without changing any results.

In condensed matter physics generalizations of both equations have played important roles as the basis of analytic models for a diversity of physical phenomena. They appear to offer an important jumping off point for further research based on well established techniques.

More recently, Matarrese and Mohayaee (2002) have presented a modification of the adhesion model that they call the *forced adhesion model*. This is based on the *forced Burgers equation*, which they transform into a random heat equation and solve using path integrals. It should be noted that this approach is in fact quite different from that of Jones (1999): Matarrese and Moyahaee use different variables and

they claim to model the self-gravity of the system, thereby avoiding Jones' external field approximation.

Menci (2002), in an approach rather similar to Matarrese and Moyahaee, also avoids the external field assumption. This is done by generalizing the simplistic gravitational terms of the classical adhesion model to a form that, it is claimed, extends the validity of the gravitational field terms. Despite the greater complexity, a solution can be achieved via path integrals.

The main shortcoming of the Jones (1999) model is indeed the assumption of an externally specified random gravitational potential field, though it is not clear that the proposed alternatives are much better. In the Jones model the intention had been to write two equations: one collisionless representing dark matter and providing the main contribution to the gravitational potential and the other collisional, representing the baryonic (dissipative) component. That program was never completed.

VIII. CONCLUDING REMARKS

A. About scaling

As we have demonstrated above, there are many scaling laws, which connect cosmological observables. The main reasons for that are the scale-free nature of gravitation and the (hopefully) scale-free initial perturbations.

The gravity scaling could, in principle, extend into very small scales, if we had only dark matter in the universe. In the real world the existence of baryons limits the scaling range from below by typical galaxy masses.

The scaling range starts from satellite galaxy distances, several tens of kpc, and it may extend up to cluster sizes, 10 Mpc; two-three decades is a considerable range. The scaling laws at supercluster distances and larger are determined by the physics of initial fluctuations.

The first scaling law characterizing the distribution of galaxies is the power-law behavior of the two-point correlation function at small scales: $\xi(r) \propto r^{-\gamma}$. Other authors try to fit the quantity $1 + \xi(r)$ to a power law $\propto r^{D_2-3}$. Obviously the previous two power laws can only hold simultaneously within the strong clustering regime, where $\xi(r) \gg 1$ and, therefore –only at those scales– the equality $\gamma = 3 - D_2$ holds. At intermediate scales ($3 < r < 20\,h^{-1}$ Mpc) the correlation dimension D_2 is ~ 2, increasing at larger scales up to $D_2 \simeq 3$, indicating an unambiguous transition to homogeneity. Moreover the statistical analysis of the galaxy catalogs permits to conclude that, within the fractal regime, the scaling is better described in terms of multifractal inhomogeneous measures rather than using homogeneous self-similar scaling laws.

Scaling of the galaxy correlation length r_0 with the sample size, $r_0 \propto R_s$, is a strong prediction for a fractal distribution. Nevertheless, this behavior is clearly ruled out by the present available redshift catalogs of galaxies. The scaling of r_0 for different kind of objects –from galaxies to clusters including clusters with different richness– has been expressed as a linear dependence of r_0 with the intercluster distance d_c. This law, however, does not hold for large values of d_c.

One successful scaling law found in the distribution of galaxies is the scaling of the angular two-point correlation function with the sample depth. In this case however, the scaling argues against an unbounded fractal view of the distribution of galaxies, supporting large-scale homogeneity.

Finally, the hierarchical scaling hypothesis of the q-order correlation function needs further confirmation from the still under construction deep and wide redshift surveys.

We have attempted here to provide an overview of the mathematical and statistical techniques that might be used to characterize the large scale structure of the universe in coordinate space, velocity space, or both, with, we hope, enough reference to actual applications and results to indicate the power of the various techniques and where they are likely to fail. Of these methods, the ones that have been used most often and so are needed for reading the current literature are the two-point correlation function (Sect. V.B), the power spectrum (Sect. VI.C), counts-in-cells and the void probability function (Sect. VI.E.3), and fractal and multifractal measures (Sect. VI.E.4). Those that we believe have the most potential for the future analysis of the very large redshift data bases currently becoming available are the Fourier methods (Sect. VI.C and Sect. VI.D), although surely the reliable determination of the two-point correlation function at large scales is still very important for understanding the large-scale structute (Durrer *et al.*, 2003).

Most of the techniques can be applied equally well to real data (in two or three dimensions) or to the results of numerical simulations of how structure ought to form in universes with various cosmological parameters, kinds of dark matter, and so forth (also in three dimensions or two-dimensional projections).

B. Future data gathering

It may well be that the 2dF and SDSS surveys are the last great redshift surveys for some time to come. They have yielded a phenomenal amount of new information which we have hardly had time to fully digest. It is not clear what extra information another million redshifts might yield: long term funding issues may prevent us from ever seeing that. However, the future may well lie in the direction of deeper surveys probing those times when the galaxies themselves were forming and the large scale structure was coming into existence.

A number of such surveys are currently under way: 2MASS, COMBO17, GOODS, DEEP2, CADIS, and the recently funded ALHAMBRA. With these we will be able to confront our models with real data, but only provided we can filter out the effects of galaxy evolution which will affect sample selection and data interpretation (particularly if there are luminosity dependent effects).

C. Understanding structure

We have tried and tested a number of descriptors of the galaxy distribution with varying amounts of success. The task has been helped by ever-growing data sets, but it is nevertheless becoming clear that a somewhat different approach may be required if we are to improve substantially on what we understand now.

What different approaches might we take? Our visual impression of large scale structure is that it is dominated by voids, filaments and clusters. This suggests that instead of looking at sample-wide statistical measures such as correlation functions, we might try to isolate the very features that strike us visually and examine them as individual structures. Much effort has already been devoted to isolating "clusters" of galaxies, but there are currently few, if any, methods available for isolating either voids or filaments.

Wavelet analysis and its generalizations such as Beamlets and Ridgelets may prove useful in identifying these structures (Donoho *et al.*, 2002). Other nonlinear analysis methodologies exist but have not been tried in this context. The fact that galaxies (or points in a simulation) provide a sparse Poisson sample of the underlying data complicates the application of what might otherwise be standard methods.

The power of having a clear mathematical descriptor lies in being able to unambiguously identify and study specific objects. This in turn provides a tools for confronting simulations with data.

D. About simulations

Ever since the first simulations by Aarseth, Gott and Turner we have gazed upon and admired simulations looking "as good as the real thing". We were impressed by the gravitational growth of clustering and we were impressed by the fact that the two-point correlation function exhibited a power law of approximately the right slope.

Subsequent developments explored the dependency of the results on initial conditions and extended significantly the range of length scales over which we could apply our value-judgements. There has also been a clearer discrimination between dark matter (the stuff of simulations) and the luminous matter (the stuff we observe). To this has been added exceptional computer graphics to render the simulations as "observed samples". They look as good as the real thing.

Several caveats apply. First, simulations provide three space and three velocity coordinates for each mass point at each time. Data provide two (angular) space coordinates and a redshift, which is made up of two terms, one proportional to the third spatial coordinate (distance) and one representing motion of the point (galaxy or cluster) relative to uniform cosmic expansion. These can be separated only within some model of what real (rather than N-body) clusters ought to be doing in the way of a Virial theorem or some other way of parcelling out potential and kinetic energy among the mass points.

Second, between the simulations of what the (mostly dark) matter is doing and data on what luminous galaxies are doing lies all of what one might call gaseous astrophysics (or even gastrophysics). The intermediate territory includes inflow of baryons into the potential wells, star formation and wind energy input, supernovae (which add both kinetic energy and heavy elements, which change how gas cools and condenses), galactic winds, on-going infall into the wells, systematic gas flow within galaxies, shocking of baryons plus heating and/or triggered star formation when halos interact, collide, and merge, energy input from black hole accretion, and so forth. Most of these currently defy real calculation and are represented by parameters and proportionalities. Thus the statement that some particular set of cosmological parameters, initial conditions, and prescriptions for star formation evolve forward in time to "fit the data" is not equivalent to being able to say that this is the way nature did it.

E. Where we stand on theory

The evolution of cosmic structure is a complex nonlinear process driven mainly by the force of gravity. The simplicity of the underlying driving mechanism, Newtonian attraction, and the fact that we see simple power law scaling, leads us to believe that the process of how large scale cosmic structure is organized can be understood. What is missing is a clear methodology for this, and it is certain that we shall to borrow tools and methods from other branches of physics. This is of course easier said than done since the driving force, gravity, has infinite range and is always attractive.

Two approaches look promising at this time. There is the numerical Renormalization group simulations of Peebles and Couchman. Then there are the analytic models: the Void Hierarchy models of Sheth and van de Weygaert and the Peak Patch model of Bond and Myers. The Random Heat Equation model of Jones and the Schrodinger Equation approach of Widrow and Kaiser remain to be fully evaluated.

F. And finally ...

We have good reason to believe that our data samples are now good enough to unequivocally allow an unambiguous description of the clustering of galaxies in the Universe. This description is entirely consistent with the view of the Universe as a whole that has emerged from the theoretical and observational research of the 20th. century. There are many details to fill in and there is much left to understand. The details will come with future observational projects and the understanding will come with further exploitation of cross-disciplinary physics. It is the existence of scaling laws in the galaxy distribution that provides us with a ray of hope that it is possible to do more than merely models the growth of cosmic structure: we may be able to understand it.

Arguably the single greatest surprise is how relatively well even rather simple models appear to reproduce the hard-won data.

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