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THE LUMINOSITY FUNCTION OF GALAXIES

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1. THE RELEVANCE OF THE OPTICAL LUMINOSITY FUNCTION

1.1 *The General vs. Specific Functions*

The *optical* luminosity function (LF) of galaxies is a probability distribution $\varphi_T(M)$ over absolute magnitude M for galaxies of any specified Hubble type T. Summed over all types, it is usually called the *general* LF $\varphi(M)$ (sometimes also labeled the *universal* function). A distinction is often further made between the general function for *field* galaxies and for *cluster* galaxies because of the differing morphological mixes of these two environments (Hubble & Humason 1931, Morgan 1961, Abell 1965, Oemler 1974).

We emphasize throughout this review that the concept of the general

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LF can no longer be maintained with sufficient precision for many cosmological calculations because the relative frequencies of the Hubble types depend so strongly on the environmental density. Dressler's (1980) density-morphology relation for clusters has been shown to apply to sparse groups (Bhavsar 1981, de Souza et al. 1982) and to the general field (Postman & Geller 1984), embracing density ratios of cluster to field as high as 10^6. Since the specific functions $\varphi_T(M)$ differ in *shape* for different T, their sum over T—the general function—cannot have a universal shape for all environmental densities. At best, such a general function can refer only to an average morphological mix at some average density that applies to galaxies in a particular sample. This theme of increasing detail, going from an average general function to particular sums of specific functions $\varphi_T(M)$ in different environments is the principal one of this review.

The fact that the concept of a general function is now inadequate is not trivial in its consequences. Accurate knowledge of the LF is required for many calculations in cosmology. Integrations over space and time must be made to predict various observable distributions. These functions, often at the core of observational cosmology, either test world models or are important in the search for secular evolution in the look-back time. It is here that the LF forms a basic ingredient in *practical* cosmology, in addition, of course, to its deeper significance concerning the physical characteristics of galaxies. In this latter role, the LF holds clues to the formation and evolution of galaxies and of clusters, especially evident from the consequences of the type-density relation [see Dressler (1984) for a review].

1.2 *Some Examples Where $\varphi(M)$ is Needed*

Many uses of the general differential luminosity function (see Section 2 for definitions) are mentioned by Schechter (1976) in the introduction to his influential paper. These include (*a*) the conversion of the observed (projected) angular correlation function to the spatial (three-dimensional) covariance function; (*b*) the calculation of the luminosity density averaged over cosmologically interesting volumes; (*c*) the determination of selection effects on particular parameter averages in samples chosen by apparent magnitude (Schechter notes only the one example of the mean binding energy of pairs of galaxies, but every calculation of a true distribution, recovered from any particular observed flux-limited sample, is similar); and (*d*) the estimation of the number of absorbers at different redshifts and different cross sections to produce the "Lα forest" in quasi-stellar objects, etc.

To illustrate, we now examine four such problems in more detail so as to emphasize the importance of $\varphi(M)$ in practical cosmology.

1.2.1 THE MEAN LUMINOSITY DENSITY A first estimate of the luminosity density of galaxies can be made by combining the galaxy count numbers $N(m)$ with some value of the *average* absolute magnitude, say M^*, in the Schechter function, the analytical formulation of Abell's (1962, 1964, 1972) description of the two asymptotic behaviors of $\varphi(M)$ at the bright and faint end, separated at the M^* "break." Bright-galaxy counts, fitting only data in the southern Galactic hemisphere, give (Sandage et al. 1972)

$$\log N(m) = 0.6m - 9.31, \tag{1}$$

where $N(m)$ is the number of galaxies per square degree brighter than m. Assigning various average absolute magnitudes to the types of galaxies counted gives the volumes surveyed by galaxies in the interval $m-0.5$ to $m+0.5$. The number of galaxies in this same magnitude interval calculated from Equation 1, multiplied by the assumed average luminosity per galaxy, gives luminosity densities of $1.1 \times 10^8 L_{B\odot}$ Mpc^{-3} if $M^*_B = -19$, 6.8×10^7 in the same units if $M^*_B = -20$, 4.0×10^7 if $M^*_B = -21$, etc. [The M^* value calculated by Tammann et al. (1979, their Table 2) from the *Revised Shapley-Ames Catalog* (Sandage & Tammann 1981) was -20.7 for the total sample, assuming a Hubble constant of 50.]

The more detailed, but much more complicated, calculations of the luminosity density using the methods for finding the distribution of M (i.e. the luminosity function) discussed in Section 3 have been made by many authors; they have been reviewed by Huchra (1986). Most values are within $\pm 10\%$ of

$$L = 8 \times 10^7 (H/50) L_{B\odot}\,\mathrm{Mpc}^{-3}, \tag{2}$$

corrected for internal absorption and averaged over what Yahil et al. (1979, 1980) considered to be a global mean density. The consequence of Equation 2, combined with the closure density of $3H^2/8\pi G$, is that

$$\Omega_0 = \rho/\rho_{\mathrm{crit}} = 1.2 \times 10^{-3} \left(\frac{M}{L}\right)\left(\frac{H_0}{50}\right)^{-1}. \tag{3}$$

If $H_0 = 50$, M/L must equal ~ 1000 for $\Omega_0 = 1$.

1.2.2 PREDICTION OF THE REDSHIFT DISTRIBUTION IN VARIOUS MAGNITUDE INTERVALS Galaxies that appear within an apparent magnitude interval $m \pm \frac{1}{2}dm$ are spread in distance, and therefore in redshift, according to their distribution of absolute magnitudes $\varphi(M)$. If $\varphi(M) = \delta(M)$, the galaxies that contribute to the interval dm at m are all within a distance range dr at r given by

$$\log r = 0.2(m - M + 5), \tag{4}$$

and are therefore within the redshift interval

$$dv = H_0\,dr, \tag{5}$$

where H_0 is the Hubble constant. When $\varphi(M) \neq \delta(M)$, but rather has a *distribution* of absolute magnitude, the number of galaxies in the magnitude range (m_1, m_2) at velocity v in interval dv in solid angle ω is given by

$$N(v)\,dv = \omega v^2\,dv\,H_0^{-2} \int_{m_1}^{m_2} \int_{-\infty}^{+\infty} \varphi(M) D(10^{0.2(m-M-5)})\,dm\,dM, \tag{6}$$

where D is the number of galaxies per cubic megaparsec at the distance $r = 10^{0.2(m-M-5)}$. Equation 6 can be used, for example, to calculate the expected redshift distribution of a complete sample of galaxies between, say, apparent magnitudes $m-0.5$ and $m+0.5$. The equation assumes Euclidean geometry and is valid therefore for low velocities ($z \lesssim 0.5$). Proper volumes for various q_0 values must be used in the general case (cf. Section 1.2.4).

An example of predicted velocity distributions for galaxies between $m = 10$ and $m = 11$ up to $m = 14$ to $m = 15$ using the general luminosity function given by Tammann et al. (1979; hereinafter TYS) has been calculated by Schweizer (1987, Figure 12). The observed redshift distributions for very faint galaxies have been summarized by Ellis (1987) and Koo & Kron (1987) for two narrow pencil-beam surveys to $B \sim 21$ and $B \sim 22$, respectively. The decided nonuniformity in both distributions is because the surveys cut through the boundaries of sheets and voids along the line of sight. An accurate calculation of the expected envelope of the distribution requires knowledge of $\varphi(M)$, the K correction (see Section 2), and luminosity evolution at each look-back time.

1.2.3 PREDICTED SURFACE DENSITY OF dE DWARFS THAT WILL BE BRIGHTER THAN APPARENT MAGNITUDE *m* Biased galaxy formation requires that the giant-to-dwarf ratio be a function of the mean density. Faint galaxies should exist in the low-density regions, but giants should be absent. On the other hand, if dwarf galaxies can only form as satellites of giants, the giant-to-dwarf ratio should *not* depend on environmental density. A search for dwarf galaxies in the general field (Binggeli et al. 1988) can address this problem of the shape of $\varphi(M)$ depending on density. Predictions of the expected surface density of dwarfs indicate what such a survey might find.

The number of galaxies per square degree that should be present in the apparent magnitude interval dm at m, contributed from the absolute luminosity interval M_1 to M_2, is

$$A(m)\,dm = \frac{4\pi}{41{,}253} \int_{M_1}^{M_2} r^2 \varphi(M)\, D(r)\,dr\,dM, \tag{7}$$

where m, M, and r are related by $m-M+5 = 5 \log r$. [The number of square degrees in the sky is $4\pi(180/\pi)^2 = 41{,}253$.]

The most illuminating way to solve Equation 7 is to replace the integral by a sum over discrete volume segments defined by inner and outer distances separated by $\Delta\log r = \log r_2/r_1 = 0.2$. This gives intervals of 1 mag in $m-M$. If $\varphi(M)D(r)$ is tabulated for 1-mag intervals (such that $M+0.5$ and $M-0.5$ are the boundaries of the tabulation), $A(m)$ will be the surface density of objects at m in the 1-mag interval $m-0.5$ to $m+0.5$. This procedure is the m, $\log\pi$ method of solving Equation 7, originally due to J. C. Kapteyn and to F. H. Seares (cf. Bok 1936, Mihalas & Binney 1981).

Applying Equation 7 with $\varphi(M)$ from TYS (their Figure 3) and integrating over the dwarfs defined as galaxies fainter than $M = -15$ gives a series of $A(m)$ values for various assumptions of the slope of $\varphi(M)$ at the faint end. For exponential increases in $\varphi(M)$ given by $\log\varphi(M) = \text{constant} + am$ (fitted to the bright-end shape and normalization of TYS), summing the $A(m)$ values to obtain $N(m)$ gives the predicted number of dwarfs brighter than $m = 17.5$ and 18.5 per square degree whose absolute magnitudes are between -15 and -8 (Table 1). The third line of Table 1 gives the ratio of the number of dwarfs to the total number of galaxies (of all luminosities) and shows that only a few percent of a complete surface survey of galaxies are expected to be dwarfs.

A more useful calculation of the expected numbers of dE and Im types taken separately requires knowledge of the *specific* luminosity function $\varphi_T(M)$ for each of these types (Section 5).

1.2.4 THE COSMOLOGICAL N(m) TEST Galaxy counts to faint magnitudes give $A(m)$ and hence $N(m) = \int A(m)\,dm$. These observational data can be compared with calculated $A(m)$ values using an equation similar to Equation 7. But there now is the complication of spatial curvature for the volume element. Also, the Mattig (1958) relation between m, M, and r must be used rather than $m-M+5 = 5\log r$. Luminosity evolution in the look-back time can be included by making $\varphi(M)D(r)$ a function of r

Table 1 Expected $N(m)$ per square degree for dwarfs fainter than $M = -15$

	Slope values		
$m_B \leq$	0.2	0.3	0.4
17.5	0.1	0.5	2.6
18.5	0.5	2.0	10.2
Percent of total	0.3	1	5

(or redshift, meaning time). Hence, the look-back time as a function of geometry must also be known. The K-correction (Section 2) also becomes very important and can be included in the $\varphi_T(M, z)$ relations for various galaxy types.

No details of these complicated calculations have yet been given either in the literature or in textbooks, but the concepts are straightforward using the version of Equation 7 that takes non-Euclidean geometry into account.

Results (but not the details) of such calculations, with and without evolution, are given by Peterson et al. (1979), who also provide references for the pre-1979 literature. A review by Ellis (1987) gives more recent results.

1.3 *History of Determining the General Luminosity Function*

Soon after the general acceptance of nebulae as galaxies, and following Knut Lundmark's studies of resolved stars in M33, E. J. Öpik's dynamical determination of the distance to M31, and Edwin Hubble's discovery of Cepheids in M31, M33, and NGC 6822, Hubble began a study of the spread of M for galaxies. It soon became evident that the scatter of the velocity–apparent magnitude relation (the Hubble diagram) *was small* for the Shapley-Ames galaxies studied by Humason (1936), which included a few velocities then known from the beginning of N. U. Mayall's Lick Observatory program. On the assumption of a linear velocity-distance relation, this meant that the spread in M for these galaxies was also small. This led Hubble (1936a,b,c) to claim that a Gaussian distribution held for $\varphi(M)$ with $\sigma = 0.84$ mag. This distribution was obtained from the residuals of the velocity-magnitude relation and was seemingly confirmed by an independent method based on the apparent magnitudes of the brightest resolved "stars," which later were found to be mostly H II regions.

Hubble's Gaussian form was later argued to be incorrect because of neglected selection effects (Zwicky 1942, 1957, 1964), with the true (complete) LF being an increasing exponential to very faint dwarfs. Holmberg (1950) showed that the addition of the known faint dwarfs in the Local Group indeed skewed Hubble's symmetrical Gaussian on the faint side. Abell's (1962, 1964, 1972) study of the LF of clusters was the first convincing evidence that dwarf galaxies have an exponentially rising LF, contrary to the bounded $\varphi(M)$ on both the bright *and faint* end for spirals, S0, and high-surface-brightness elliptical galaxies. The publication by Reaves (1956) of data on dwarfs in the Virgo cluster, anticipated by Baade (1950) and discussed earlier by C. D. Shane (quoted in Baade 1950), began to convince Zwicky's critics that his 1942 conjecture for an exponential faint tail to $\varphi(M)$ was correct. Krupp (1974) later published the most comprehensive cluster LF to date [following the important work of Rood (1969) in the Coma cluster] based on Palomar Observatory 5-m reflector plates. Modern work in the Coma cluster by Thompson & Gregory (1980) will soon be superseded by an even more comprehensive survey by the same authors, again in Coma.

Holmberg's (1969) study of companions to field giants suggested (only a few redshifts were available) that the general LF outside the great clusters also has an exponentially rising faint end rather than a bound as in Hubble's first formulation.

In recent years it has become clear that both Hubble and Zwicky were correct for the types of galaxies each discussed. Hubble's list was almost exclusively high-surface-brightness galaxies, whereas Zwicky's faint rising exponential tail almost entirely contained galaxies of low surface brightness, none of which were in Hubble, Humason, or Mayall's early redshift program. The shapes of the two luminosity functions, (nearly symmetric Gaussians for the high-SB cases, ever-increasing exponentials for low-SB cases) provide the first and most obvious example of a difference in the specific $\varphi_T(M)$ functions for different types. Other, more modern examples are the first-ranked cluster E galaxies, which have an exceptionally small dispersion of $\sigma \simeq 0.3$ mag and a nearly Gaussian $\varphi(M)$ (Sandage 1972, Sandage & Hardy 1973, Schneider et al. 1983), and ScI spirals, which have $\sigma \simeq 0.7$ mag and also a nearly Gaussian $\varphi(M)$ (Sandage & Tammann 1975).

Studies of the general LF of the field were hampered early on by the sparseness of the redshift data for the samples studied. Early determinations of the bright end of the field function (van den Bergh 1961, Kiang 1961) depended on the redshift catalog of Humason et al. (1956; hereinafter HMS), with magnitudes from Pettit (1954) corrected for aperture effect by HMS. This redshift sample was not appreciably increased until the late 1970s. The most comprehensive studies before the large new redshift surveys began are those of Shapiro (1971) and Christensen (1975). These studies were innovative in that they divided the discussion into morphological types, which was the beginning of our present emphasis on the specific $\varphi_T(M)$ function for each type.

An important review by Felten (1977) of the work on the general field LF summarizes nine independent studies of the shape of $\varphi(M)$ for "field" galaxies and pays particular attention to the absolute normalization. This is treated as the density function in Section 2, and it normalizes $\varphi(M)\langle D(r)\rangle$ into units of the number of galaxies at $M \pm \frac{1}{2}dM$ per cubic megaparsec.

Felten's review was the last to summarize work before the new data from comprehensive redshift surveys began to be analyzed. The first of these new data sets was the completion of redshift coverage of Shapley-Ames galaxies on those galaxies not measured by Humason and Mayall (Sandage 1978). These data were combined with all others in the literature to produce the *Revised Shapley-Ames Catalog* (Sandage & Tammann 1981; hereinafter RSA), which was analyzed by TYS, accounting for selection effects. A mean total (general) LF was derived that was separated into various spiral types and van den Bergh luminosity classes. The result was compared with the LF for ellipticals. From this large sample, the average given in Equation 2 was derived, which is similar to the values of the luminosity density determined previously (reduced to the same value of H_0) from the many studies already referenced.

Redshift surveys to fainter magnitudes also began to be analyzed. A result by Kirshner et al. (1979) for their special regions, where redshift distributions to fixed magnitudes had been obtained, was compared with the so-called universal Schechter (1976) general function. Davis & Huchra (1982) later initiated the analysis of the field galaxy sample of the Center for Astrophysics (CfA) redshift survey, complete in various regions to $m \simeq 14.2$.

A new development was the study by Tammann & Kraan (1978), who analyzed a volume-limited sample rather than one that was flux-limited such as the RSA or the CfA survey (Huchra 1985). They obtained separate luminosity functions for various Hubble types (i.e. the specific $\varphi_T(M)$ functions). The analysis was based on their distance-limited catalog for galaxies within a corrected redshift of 500 km s^{-1} (Kraan-Korteweg & Tammann 1979).

Finally, still commenting on studies of the field LF, we note the new methods of analysis introduced by Turner (1979), Kirshner et al. (1979), Sandage et al. (1979), and Choloniewski (1985, 1986, 1987) to deal with selection effects.

Studies of the LF of *clusters* is inherently easier than for field galaxies because the problem of knowing individual distances is circumvented. The work of Abell and Krupp already has been mentioned above. Rood's (1969) work on the Coma cluster initiated the modern methods of cluster photometry. Early examples are the studies by Oemler (1974), Godwin & Peach (1977), Dressler (1978), and Kraan-Korteweg (1981) of the Virgo cluster, which are the first modern deep studies of this cluster.

Most of the studies mentioned in this section were concerned with the general LF integrated over all Hubble types and environmental densities. The remainder of this review sets out in a new direction, hinted at in some of the earlier referenced papers—namely, the work with the individual specific functions $\varphi_T(M)$.

2. DEFINITION

2.1 *Luminosity Function and Density Function*

Let $\nu(M, x, y, z)$ denote the number of galaxies lying in volume dV at (x, y, z) that have absolute magnitudes between M and $M+dM$. On the assumption that galaxian magnitudes are not correlated with spatial location, one can write

$$\nu(M, x, y, z)\, dM\, dV = \varphi(M)\, D(x, y, z)\, dM\, dV, \qquad 8.$$

where

$$\int_{-\infty}^{+\infty} \varphi(M)\, dM = 1. \qquad 9.$$

$\varphi(M)$ gives the fraction of galaxies per unit magnitude having absolute magnitudes in the interval $(M, M+dM)$ and is called the *luminosity function*. $D(x, y, z)$ gives the number of galaxies (of all magnitudes) per unit volume at (x, y, z) and is called the *density function*. φ and D should be viewed as probability densities, which in practice are approached and represented either by (nonparametric) histograms or by (parametric) analytical forms.

If Equation 8 is valid for a sufficiently large portion of the Universe, or for sufficiently many samples of galaxies, $\varphi(M)$ can be called the *universal* luminosity function of galaxies. This is clearly an approximation. In reality one expects that φ does somehow depend on the location, i.e. on the environment from which the galaxies are sampled. The question of universality of $\varphi(M)$ is discussed in Section 4.3 ff. Systematic differences of $\varphi(M)$ with respect to type and environment, which are discussed in Sections 5 and 6, lead us to reject universality in the above sense; Equation 8 is subsequently revised in Section 6.3.

The present definition of the luminosity function of galaxies, as expressed by Equations 8 and 9, is identical to that used in stellar statistics (von der Pahlen 1937, Mihalas & Binney 1981). It should be noted that the conventional definition of the galaxian $\varphi(M)$ was different during the previous decades. The usual ("classical") method to determine the luminosity function of field galaxies (outside of rich clusters) was based on the assumption that $D(x, y, z) = \langle D \rangle =$ constant (see Section 3.2), which allows the product $\varphi(M) \cdot \langle D \rangle$ to be discussed as *one* function; ever since van den Bergh (1961) and Kiang (1961), this product has been tagged with the label "luminosity function $\varphi(M)$," most recently in the review of Felten (1985).

Consequently the "luminosity function" has been given the units of density (number of galaxies per magnitude per cubic megaparsec). The drawback of this definition is the creation of an artificial dichotomy between field and cluster samples. Clusters of galaxies, where $D \neq$ constant is obvious, could strictly not have a luminosity function. In order to distinguish the function $\varphi(M)\langle D\rangle$ from $\varphi(M)$, Schechter (1976) has introduced for the latter the term "luminosity distribution." Starting with Sandage et al. (1979) and Kirshner et al. (1979), the assumption that $D =$ constant for field galaxies has been dropped. The general inhomogeneity of the distribution of galaxies is now widely acknowledged, and all new methods used to derive the luminosity function of field galaxies aim at a clear separation of φ and D (see Section 3.2, Table 1). A redefinition of $\varphi(M)$ along the lines of stellar statistics (Equations 1, 2) is therefore most desirable at present. The mean density D, averaged over a significant portion of the observable universe, remains of course a most important quantity for cosmology, but there is no reason why it should be built into the luminosity function (provided that the notion of a universal shape for $\varphi(M)$ makes any sense at all). The discussion of D is consequently left out of the present review.

The normalization of $\varphi(M)$ to unity by integrating over all magnitudes (Equation 9) is difficult in practice because any sample of galaxies is complete, or has good statistical weight, only to a certain limiting magnitude $M_{\lim}$. The ideal case, where the faint end of $\varphi(M)$ goes to zero at a magnitude $M' \leqq M_{\lim}$ is at present applicable only to certain types of galaxies that are sampled nearby (cf. Section 5). In general, φ not only is nonzero but is growing exponentially at $M_{\lim}$, making such a normalization infeasible; any extrapolation of $\varphi(M)$ to fainter magnitudes by an analytical model will diverge. A way to avoid this divergence would be to go to the luminosity (L) representation of the luminosity function, transforming $\varphi(M)$ into $\varphi(L)$ and setting

$$\int_0^\infty \varphi(L)\, dL = 1, \qquad 10.$$

which for physical reasons must always converge. However, we wish to keep the magnitude representation, since $\varphi(M)$ is closer to the observations than is $\varphi(L)$. An obvious and practicable way to normalize $\varphi(M)$ is to restrict the discussion to galaxies brighter than a certain arbitrary absolute magnitude $\tilde{M}$, in which case Equation 9 is replaced by

$$\int_{-\infty}^{\tilde{M}} \varphi(M)\, dM = 1. \qquad 11.$$

D in Equation 8 is then the density of galaxies that are brighter than $\tilde{M}$.

$\tilde{M}$ may be different for different samples. Future work will push $\tilde{M}$ toward fainter and fainter limits until the ideal normalization of Equation 9 can be realized.

It should be noted that the normalization is not a principal problem for the present concept. A normalization of $\varphi(M)$ is needed only for the discussion of the density function D, which, by virtue of the adopted separation, is not a subject of this review. [Densities are only discussed where they have fundamental consequences for $\varphi(M)$, such as in the context of the morphology-density relation (see Section 6).] For the discussion of $\varphi(M)$ alone, no normalization is required because it is a probability distribution. Therefore, *any* $\varphi(M)$, *whether normalized or not, is called here a luminosity function.* The luminosity functions of different samples can then be compared by their *shape* (i.e. the shape *is* the luminosity function). What matters only is that $\varphi(M)$ is decoupled from the density function.

$\varphi(M)$ is sometimes called the *differential* luminosity function, which should be distinguished from the *integrated* (or *cumulative*) luminosity function $\Phi(M)$, defined as

$$\Phi(M) = \int_{-\infty}^{M} \varphi(M')\, dM'. \qquad 12.$$

$\Phi(M)$ is less frequently used than $\varphi(M)$; it tends to conceal an intuitive interpretation of the information available for the fainter galaxies. In what follows, unless otherwise stated, LF is always meant to designate the *differential* luminosity function $\varphi(M)$.

2.2 *Magnitudes*

It is of the utmost importance to pay attention to the exact definitions of, and the corrections to, the apparent magnitudes used by various authors to derive the LF. The following parameters must be specified:

1. The passband of the magnitudes. Usually total blue magnitudes B_T are used as defined in *Second Reference Catalog of Bright Galaxies* (de Vaucouleurs et al. 1976; hereinafter RC2). Frequently, magnitudes must be used that only approximate the B_T system, as for instance Zwicky's magnitudes (Zwicky et al. 1961–68). Infrared workers sometimes follow the example of radio astronomers by using fluxes rather than magnitudes (e.g. Lawrence et al. 1986).

2. The Galactic absorption, which must be corrected for. This correction can be achieved by specific absorption determinations in relevant fields, by following the precepts of the RSA (Sandage & Tammann 1981),

by using the maps of Burstein & Heiles (1982), or by any other appropriate method.

3. The internal absorption, which may or may not be corrected for. The internal absorption of E and S0 galaxies is generally neglected. The exact correction for spirals is not well known. The RC2 gives the absorption to face-on orientation, assuming the same absorption for all spiral types and somewhat higher values for Im's. Following Holmberg (1958) with slight modifications, the RSA applies inclination-dependent corrections for the total internal absorption, with the highest absorption corrections for Sb's ($A^{i}_{B} = 1.33$ mag for an edge-on Sb galaxy). IRAS data seem to confirm that the internal absorption peaks for Sb's (de Jong & Brink 1987).

Most published LFs have used magnitudes that are uncorrected for internal absorption. Corrected magnitudes M^{i} were used by Kiang (1961), Tammann et al. (1979), and Kraan-Korteweg (1981). Deciding which procedure is preferable depends on the application of the LF. If spirals are randomly oriented (cf. Djorgovski 1987), uncorrected LFs should be used for the interpretation of galaxy counts and of the cosmic background light. On the other hand, only the LF $\varphi(M^{i})$ gives correct results if, for example, the total hydrogen consumption or the energy output of a sample of galaxies is required, or if physically meaningful mass-to-light ratios are to be calculated.

Unfortunately, a bulk transformation of $\varphi(M)$ into $\varphi(M^{i})$, or vice versa, is not possible. The exact conversion depends on the specific mixture of galaxy types, which depends upon the environmental density whose average usually varies for subsets of the total sample. Furthermore, in the case of flux-limited samples the two types of LFs are also based on different parent samples; while $\varphi(M)$ considers all galaxies brighter than the limiting magnitude m, $\varphi(M^{i})$ includes in addition all inclined spirals that are brighter than *m after* the absorption A^{i} is applied. In some cases, the available data permit both $\varphi(M)$ and $\varphi(M^{i})$ to be determined (cf. Tammann et al. 1979).

4. The *K*-correction for redshift dimming, which must be applied for distant galaxies. For redshifts $z \lesssim 0.02$ the *K*-correction in optical passbands remains smaller than $0^{m}.1$ for all galaxy types (Whitford 1971, Wells 1972) and may be neglected for the LF. However, at large redshifts the *K*-correction not only becomes large but also is sensitive to the galaxy type. For instance, at redshift $Z = 0.5$ the difference in the *K*-correction may amount to $\sim 1^{m}.5$ between different types (Pence 1976, Coleman et al. 1980). If such large effects were neglected, comparison of the LFs of nearby and high-redshift galaxy samples could lead to erroneous conclusions on galaxy *evolution*.

2.3 *Selection of Galaxies*

The selection of galaxies is fundamental for the LF. Ideally, the galaxies are selected by *total apparent* magnitude. In reality, the galaxies are always drawn from catalogs that are based on *photographic* surveys. The *detection* of a galaxy on a photographic plate does not depend on total magnitude but on surface brightness. Both very compact, high-surface-brightness objects and extended, very low-surface-brightness objects have small isophotal diameters and can go undetected. Reaves (1956) and Arp (1965) were the first to draw attention to this potential selection bias by showing that the normal galaxies detected so far populate a narrow strip in the magnitude–log diameter diagram. Galaxies of very low surface brightness may be entirely missed. That such galaxies exist is known from the local dwarf spheroidals (Fornax, Sculptor, et al.), which were detected only because they are sufficiently nearby to be resolved into stars. The possible consequences for the LF are clear: The LF derived from any given sample will usually refer only to normal, easily visible galaxies. A population of luminous galaxies of low surface brightness may go unnoticed, although the LF nominally refers to *total* magnitudes.

This "tip-of-the-iceberg" bias has been emphasized by Disney (1976). Disney & Phillips (1983) have quantified the visibility of galaxies as a function of central surface brightness and have shown how galaxies of "normal" surface brightness are automatically preferred in any realistic procedure of galaxy sampling. Consequently, these authors have suggested that the "bivariate brightness distribution" be used instead of the (one-dimensional) LF. This is the distribution of galaxies in total magnitude and surface brightness. The recent large-scale photographic survey of the Virgo cluster (Binggeli et al. 1985; hereinafter BST), which should have picked up all Virgo galaxies above a surface brightness threshold of 25.5 B mag arcsec^{-2} would appear to be a suitable basis to explore the bivariate brightness distribution of galaxies (cf. also Phillips & Disney 1986). A class of very extended, very low-surface-brightness objects was indeed discovered in the course of this survey, some of which are as bright as $M_{B_T} \sim -17$ (Sandage & Binggeli 1984). The dominant feature in the magnitude–surface brightness diagram, however, is a strong correlation between absolute magnitude and surface brightness for galaxies fainter than $M_{B_T} \sim -19$ and of the type dE or Sd-Im (Caldwell 1983, Binggeli et al. 1984, Binggeli 1986, Bothun et al. 1986). This would suggest that most of the galaxies that are missed because of their low surface brightness are also of low total brightness. Galaxies *are* detected by surface brightness, but (thanks to this relation) they are generally also selected by total absolute magnitude.

However, this simple view is about to be challenged by Impey et al. (1987), who, by using Malin's technique of photographic contrast enhancing, have detected many objects in the Virgo cluster and elsewhere that have a central (peak) surface brightness below 26 B mag arcsec^{-2} and that accordingly went undetected by BST. Furthermore, in the course of their survey centered on the Virgo cluster core, these authors (Bothun et al. 1987) accidentally discovered that one of their objects with a very low central surface brightness of 27 B mag arcsec^{-2} is a distant spiral with a total magnitude of $M_{B_T} \approx -22$ (!). This brings new vigor to the question of how many bright galaxies lie hidden below the conventional photographic detection limits (Disney & Phillips 1987)—and hence have been missed by all conventional LF studies to date. The answer must await completion of deep surveys like Impey et al.'s. In any case, the galaxy discovered by Bothun et al. is clearly a very rare type. Until its nature and relation to "normal" galaxies are better known, we cannot worry about the expected incompleteness of the LF of galaxies due to this new type of galaxy. Yet it is important to be aware that the LF does use total magnitude for objects that are never strictly selected by total magnitude.

3. METHODS TO DETERMINE THE LUMINOSITY FUNCTION

Galaxies are usually divided into "cluster galaxies" and "field galaxies." A "cluster galaxy" is a member of a (rich) cluster that is representative of the clusters listed in, for example, Abell's (1958) or Zwicky et al.'s (1961–68) catalogs of clusters. As "field galaxy," one can then simply declare as such every galaxy that is not lying in a (rich) cluster; groups of galaxies thus become part of the "field." This conventional definition of a field galaxy (e.g. Felten 1977) is adopted here. The distinction between clusters and field is natural in the context of the luminosity function because the methods used to derive a LF for a cluster and for a field sample are fundamentally different. In the former case, all galaxies are at the same distance; in the latter, individual distances must be known (groups of galaxies excepted). "Cluster" and "field" also denote two basic density environments of galaxies, whose LFs cannot be expected to be the same a priori. Therefore the "cluster" and the "field" LFs are treated separately in this and the following section. All methods used to determine the LF discussed here are based on the assumption that the LF does not depend on galaxian position (within the cluster, or in the field). This means that φ and D can be separated, as expressed by Equation 8. This conventional approach is challenged in Section 6, where a general "LF-density relation" is proposed.

3.1 *Cluster Galaxies*

Because all cluster galaxies are at the same distance, the apparent magnitudes m, after appropriate binning, are used directly to give $\varphi(m)$ as a histogram. If the sampling has been proper, the distribution will be complete to a certain limiting magnitude m_{lim}. Scaling by the distance modulus of the cluster (inferred, for example, from the redshift) transforms $\varphi(m)$ into $\varphi(M)$ and m_{lim} into M_{lim}. The difficulty is to *identify* the cluster members among the great number of galaxies that are always projected into the same area of the sky.

Ideally, many of the cluster members betray themselves from their morphological types. Such a morphological sampling has proved possible for the nearby clusters in Virgo (BST, Sandage et al. 1985; hereinafter SBT) and Fornax (Ferguson & Sandage 1987), which have recently been mapped with high angular resolution. The method is based on the galaxian characteristics of surface brightness at a given magnitude, and resolution into H II regions and associations, both of which can be used as *relative* distance indicators (BST). For example, intrinsically faint members of a cluster [with the exception of blue compact dwarf galaxies (the BCDs)] are of low surface brightness, whereas typical (intrinsically bright) background galaxies have high surface brightness. Only early-type giant (E and SO) and BCD galaxies are difficult to identify as cluster members, as they have high surface brightness and show little structural detail. Here one is forced to use velocity data to determine cluster membership. Moreover, both the morphological and kinematical criteria work reliably only if a cluster is fairly isolated in space. This is because there is cosmic scattering of the morphological characteristics, and the velocities are dispersed by ~ 1000 km s^{-1} around the cluster mean, making velocity discrimination unsharp unless there is a spatial void behind the cluster. Fortunately, many clusters (like the Virgo cluster) meet this requirement because they *are* in front of large voids (Sulentic 1980, Davis et al. 1982, Ftaclas et al. 1984), and hence the background contamination is minimal. Even so, there always remains a fraction of the cluster sample for which membership cannot be decided with total certainty (of the order of 10% in the Virgo cluster; BST).

If the cluster members cannot be identified on morphological grounds, the second best procedure is to sample solely by velocity. To date, this has implied a restriction to bright galaxies. Dwarf galaxies, which mostly have low surface brightness, can *only* be sampled morphologically at present. LF studies based on pure velocity sampling have been carried out for the Fornax cluster by Jones & Jones (1980) and for the Virgo cluster by Kraan-Korteweg (1981).

The normal case in cluster LF studies, however, has been a poor morphological resolution of the galaxies, with velocity data available only for a small subsample of bright galaxies. At present, this situation applies to all clusters that are more distant than Virgo and Fornax. The identification of cluster members against foreground and background galaxies (or even stars) can be achieved only in a *statistical* way for the more distant clusters. In order to clean a cluster sample of the contamination by field galaxies, one must determine the number of field galaxies in a given magnitude bin that are expected to be randomly projected onto the cluster area [usually given as log $N(m)$, where $N(m)$ is the number of galaxies that are brighter than apparent magnitude m]. These numbers are then subtracted from the uncorrected cluster LF. The log $N(m)$ values can be determined either locally by going to a field near the cluster or by adopting an average log $N(m)$ law. For the faintest magnitude bins, a similar correction must be applied for possible contamination from stars (see Austin & Peach 1974) if the survey is made with inadequate angular resolution. A type-dependent correction for field contamination so as to differentiate cluster LFs by type (Thomson & Gregory 1980) must make specific allowance for the type-density relation (cf. Section 6.1.). The most widely used log $N(m)$ curve in cluster studies (e.g. Dressler 1978, Lugger 1986, Oegerle et al. 1986, 1987) is that of Oemler (1974), who constructed a mean log $N(m)$ law from galaxy counts in several fields near his clusters. However the nonuniformity of the distribution of "field" galaxies is now generally acknowledged (e.g. Davis et al. 1982), and it must be doubted whether a field correction based on a *general* log $N(m)$ law is meaningful in all cases. There certainly are clusters where a *local* log $N(m)$ must be used (see Dressler 1978). Cluster LFs that are corrected only *statistically* for background contamination are uncertain principally at the faint end, where often the field counts to be subtracted reach the same order of magnitude as the cluster counts.

Once a corrected cluster LF is established, a useful representation is that of Schechter (1976) in the form

$$\varphi(M)\,dM \sim 10^{-0.4(\alpha+1)M}\,e^{-10^{0.4(M^*-M)}}\,dM \qquad 13.$$

corresponding to

$$\varphi(L)\,dL \sim L^{\alpha}e^{-L/L^*}\,dL, \qquad 14.$$

where α and M^* (or L^*) are free parameters. This function has superseded other analytic expressions used previously by Zwicky (1957), Kiang (1961), and Abell (1962, 1965), which are not further discussed in this review. At faint magnitudes, Equation 13 is exponential with slope $-0.4(\alpha+1)$. On the bright side, it is a double exponential that rapidly approaches zero after a turnover at a characteristic magnitude M^* (corresponding to a characteristic luminosity L^*). Because of the definition $\varphi(M)$ in Section

2, the normalization of the Schechter function is not considered. For normalization via the local density function, see Schechter (1976).

The best-fit parameters α and M^* for a cluster LF can be found by minimizing χ^2 in fitting Equation 13 to the binned magnitude data (e.g. Dressler 1978). A generalized χ^2 statistic to include the uncertainty in the background correction has been used by Lugger (1986). Alternatively, one can apply a maximum-likelihood method to the unbinned data to obtain α and M^* (Lugger 1986, Oegerle et al. 1986).

3.2 *Field Galaxies*

The determination of $\varphi(M)$ for field galaxies requires a well-defined sample whose bias properties are known. Almost always the samples are defined by an apparent magnitude cutoff m_{lim}. Unfortunately, existing galaxy catalogs are at best complete to a cutoff magnitude, which is not corrected for the direction-dependent Galactic absorption (e.g. Kiang 1976). But even to this limit, catalogs are incomplete for other reasons. The problem of low-surface-brightness galaxies has been discussed already in Section 2. Moreover, it is typical for flux-limited samples to become progressively more incomplete as the nominal value of m_{lim} is approached and to contain also, owing to magnitude errors, some fainter objects. The completeness of a catalog can be improved by adding missing objects from other sources (Kiang 1961). Alternatively, the incompleteness of a catalog can be compensated statistically by weighting each catalog entry with the magnitude-dependent incompleteness function; this function can be found by comparing the catalog under consideration with a deeper catalog, and it can be represented by an analytic distribution function (Sandage et al. 1979) of the type first used to describe α-particle range straggling (i.e. a degraded half-step function on the trailing edge, often called the Fermi-Dirac function). An elegant way to test and to correct for incompleteness is supplied by the V/V_{max} technique, originally devised for quasars (Schmidt 1968) and subsequently extended to Markarian (field) galaxies by Huchra & Sargent (1973). Here V is the sample volume between the galaxy and the observer, and V_{max} is the volume the galaxy could lie in without dropping below m_{lim} [i.e. $V_{\mathrm{max}} = V(M)$ from below]. A sample is complete at magnitude m if the average V/V_{max}, calculated for all galaxies with magnitude m, is 0.5 *if* $D =$ constant.

The absolute magnitudes of the sample galaxies must be calculated prior to the derivation of the LF. This requires distance information for every sample galaxy. The distance of field galaxies (except for very nearby ones) must be inferred from the redshift z, since no other precise method is available for all galaxy types.

For Friedmann models, the redshift in combination with H_0 and q_0

provides the luminosity distance (Sandage 1961, 1988). For small redshifts ($cz \lesssim 60{,}000$ km s^{-1}) the linear relation between the recession velocity $v = cz$ and H_0 can be used without errors of more than ~ 0.2 mag as a result of neglect of space curvature between models with q_0 of 0 and 1. For large redshifts (i.e. $z \gtrsim 0.5$) the absolute magnitude normalization does depend on q_0 (Yee & Green 1987). Observed velocities must be reduced to the centroid of the Local Group (e.g. Humason et al. 1956, Yahil et al. 1977, Richter et al. 1987). The resulting corrected velocities v_0 still carry random peculiar motions Δv, which, however, are smaller than $\Delta v \leqq 90$ km s^{-1} for field galaxies within $v_0 < 500$ km s^{-1} (Tammann et al. 1980, Richter et al. 1987). A random velocity of $\Delta v / v_0 \lesssim 0.15$ is a generous upper limit for any field galaxy; even this value would cause a random error in absolute magnitude of 0.3 mag at most and hence would broaden and flatten the LF only slightly. More serious are streaming motions of field galaxies. A Virgo-centric infall of $v_{VC} = 220$ km s^{-1} at the circle of the Local Group has a noticeable influence on the LF of field galaxies within the Virgo complex (Kennicutt 1982, Kraan-Korteweg et al. 1984). Velocity and absolute magnitude corrections have been conveniently tabulated for a self-consistent Virgo-centric infall model by Kraan-Korteweg (1986). Some effect on the LF of field galaxies with $2000 \lesssim v_0 \lesssim 7000$ km s^{-1} is also to be expected from the apex motion toward the Hydra or Centaurus cluster at $v_0 \approx 4400$ km s^{-1} (Tammann & Sandage 1985, Lynden-Bell et al. 1987). In any case, the absolute magnitudes that are derived from velocity distances may still carry small direction-dependent errors within spheres surrounding the Virgo complex until our motion toward the microwave-background (MWB) dipole is fully understood. The size of such errors, if they exist, will be less than ± 0.5 mag.

The specific choice of H_0 is irrelevant as long as only the shape of the LF is sought. However, if absolute magnitudes from velocity distances are mixed with those from directly determined distances (e.g. for Local Group members), the correct value of H_0 must, of course, be used. If LFs from different authors are compared, an adjustment for different adopted values of H_0 is necessary.

With the absolute magnitudes known, the next step is to construct $\varphi(M)$. Table 2 is an overview of the various methods that have been used to derive $\varphi(M)$ for field galaxies. It gives the references for each method and also shows whether $\varphi(M)$ is derived in a parametric or nonparametric way and what assumptions are made about the density function $D(x, y, z)$. The column "$\varphi(M)$ parametric" divides the LFs into those that have and have not been represented by an a priori analytical expression. If "yes" is indicated, the Schechter parameters α and M^* have been determined. "$D = D(r)$" indicates that spherical symmetry around the observer has

Table 2 Methods to determine the LF of field galaxies (including groups)

Method		References	$\varphi(M)$ parametric	Assumption about $D(x,y,z)$
Classical	I	Christensen 1975, Schechter 1976, Felten 1977	No	D = constant
$\Sigma 1/V_{max}$	II	Schmidt 1968 (quasars), Huchra & Sargent 1973, Felten 1976	No	D = constant (locally)
φ/Φ	III	Turner 1979 (quasars), Kirshner et al. 1979	(No)	None
	IV	Davis et al. 1980, Davis & Huchra 1982	(No)	None
Maximum likelihood	V	Sandage et al. 1979	Yes	None
	VI	Choloniewski 1985	Yes	$D = D(r)$
	VII	Marshall et al. 1983 (quasars)	Yes	$D = D(r)$
	VIII	Nicoll & Segal 1983	(No)	$D = D(r)$
	IX	Choloniewski 1986	(No)	$D = D(r)$
C-method	X	Lynden-Bell 1971 (quasars), Jackson 1974 (quasars)	No	$D = D(r)$
	XI	Choloniewski 1987	No	$D = D(r)$
Groups	XII	Holmberg 1969, Turner & Gott 1976b, Tammann & Kraan 1978	No	—
Clustering	XIII	Yee & Green 1984, 1987, Phillips & Shanks 1987	No	—

been assumed. Because this assumption is unrealistic for the field in general, it implies that a restriction of the sample to a small solid angle of sky should have been made. Methods that were originally developed for the LF of quasars, but that could in principle be applied also to field galaxies, are included in the present discussion. It is important to recall again that all methods fundamentally assume $\varphi(M)$ to be independent of D (cf. Equation 8).

Five basic methods, or families of methods, can be distinguished in Table 2.

3.2.1 THE CLASSICAL METHOD Until 10 years ago there was but one method to determine $\varphi(M)$ for field galaxies; this is now called the classical method. Its basis is the assumption that galaxies are uniformly distributed in space (D = constant). Developers and early users of the method are van den Bergh (1961), Kiang (1961), and Shapiro (1971). These authors, however, did not describe the method. Detailed recipes for the construction of $\varphi(M)$ in the classical way are given by Christensen (1975), Schechter (1976), and Felten (1977). At the heart of the method lies the calculation of the volume $V(M)$ that is effectively surveyed for galaxies of absolute magnitude M. $V(M)$ is determined by the maximum distance an object of absolute magnitude M can have and still be in the sample. The sample is limited by a fixed apparent magnitude m_{lim}, but this limit should be corrected for the direction-dependent Galactic absorption (Kiang 1976), with the excluded volume accounted for. The numbers of galaxies in bins of $(M-\frac{1}{2}\Delta M, M+\frac{1}{2}\Delta M)$ must then be divided individually by $V(M)$, giving a binned, nonparametric $\varphi(M)$. At *faint absolute magnitudes* the number of sample galaxies per bin is decreasing because the surveyed volume for them is very small owing to the bright apparent magnitude m compared with the faint absolute magnitude M sought, making $(m-M)$ very small. This is the reason that $\varphi(M)$ becomes increasingly uncertain at faint M, to the point where it becomes meaningless. This is inherent to every LF study of field galaxies from magnitude-limited samples.

For the derivation of $\varphi(M)$, Huchra & Sargent (1973) have used the V/V_{max} method. Instead of having the number of galaxies in bin $(M-\frac{1}{2}\Delta M, M+\frac{1}{2}\Delta M)$ divided by $V(M) = V_{\text{max}}$, $\varphi(M)$ is estimated by the sum $\Sigma(1/V_{\text{max}})$ over all galaxies in $(M-\frac{1}{2}\Delta M, M+\frac{1}{2}\Delta M)$. Felten (1976) has shown that the two procedures are equivalent.

3.2.2 THE φ/Φ METHOD As galaxies of different absolute magnitudes are sampled in different volumes, any spatial inhomogeneity in the distribution of galaxies will severely distort $\varphi(M)$ if it is constructed in the classical way with the assumption of homogeneity. For instance, a local density enhancement would overestimate $\varphi(M)$ for absolutely faint galaxies, which are sampled only nearby. The danger is real because of the excess of nearby galaxies in the northern sky (known already to John Herschel). But only as the general inhomogeneity of extragalactic space became obvious with the advent of appropriate redshift samples (RSA, Davis et al. 1982) has the assumption of homogeneity been dropped. [To acknowledge the necessity of this step, one may consult Figure 1 in Davis & Huchra (1982) and Figures 3 and 5 of Choloniewski (1986).] New methods for $\varphi(M)$ that do not make the assumption of homogeneity were pioneered by Turner (1979), Kirshner et al. (1979), and A. Yahil (Sandage et al. 1979). The last is a maximum-likelihood method and is discussed below. The basic idea is to consider the ratio of the number of galaxies having absolute magnitudes between M and $M+dM$ to the total number of galaxies brighter than M [in volume dV at a given location (x, y, z)]. Using Equations 8 and 12, we find this ratio to be

$$\frac{v(M,x,y,z)\,dM\,dV}{\int_{-\infty}^{M} v(M',x,y,z)\,dM'\,dV} = \frac{\varphi(M)D(x,y,z)\,dM\,dV}{\int_{-\infty}^{M} \varphi(M')D(x,y,z)\,dM'\,dV}$$

$$= \frac{\varphi(M)\,dM}{\Phi(M)} \sim d\log\Phi(M). \quad 15.$$

The main point is that the density function $D(x, y, z)$ cancels out because φ and D are assumed to be independent. The ratio of the differential to the integrated LF, φ/Φ (determined in the classical way!), is thus independent of any inhomogeneities in the distribution of galaxies. Integrating φ/Φ gives $\log \Phi(M)$, and differentiating $\Phi(M)$ back gives $\varphi(M)$. A slight variation of the method, by binning the data in equal distance intervals instead of equal magnitude intervals, has been developed and used by Davis et al. (1980) and Davis & Huchra (1982). In principle, no assumption is required about the form of $\varphi(M)$, i.e. the φ/Φ method is nonparametric. However, in practice $\varphi(M)$ has always been parametrized. Kirshner et al. (1979) have fitted directly to the corresponding ratio of the Schechter function (cf. Section 3.1, Equation 13). Davis et al. (1980) and Davis & Huchra (1982) made a (form-independent) fourth-order polynomial fit to φ/Φ, integrated analytically to find Φ, and differentiated back to φ, which was finally fitted to a Schechter function. A disadvantage of this fitting procedure lies in the large statistical noise of φ/Φ (see Figure 1 in Kirshner et al. 1979, and Figure 2 in Davis & Huchra 1982).

3.2.3 MAXIMUM-LIKELIHOOD METHODS Similar to the φ/Φ method is the method of Sandage et al. (1979), in which a quotient is again considered to make the density function cancel out. Here it is the ratio of the number of galaxies brighter than absolute magnitude M to the total number of galaxies at a given velocity v (i.e. distance). This is simply the probability $P(M, v)$ that a galaxy at v is brighter than M. The LF $\varphi(M)$ cannot, however, be directly determined from $P(M, v)$ but has now to be modeled by an analytical expression with parameters to be fixed by a maximum-likelihood technique, namely by maximizing the product L (=likelihood) of the differential probability densities $(\partial P/\partial M)$ taken at all (unbinned) data points (M, v) of the sample. The calculation of $P(M, v)$ also requires knowledge of the sample incompleteness. The explicit correction that Sandage et al. (1979) made for incompleteness (discussed above) can easily be incorporated into the calculation. By maximizing the likelihood product L, the Schechter parameters α and M^*, as well as the parameters m_{L} and Δm_{L} of the incompleteness function $f(m)$, were then found simultaneously. A somewhat different completeness function was used earlier by Neyman & Scott (1974), who were among the first to use the maximum-likelihood technique in galaxy statistics.

In contrast to the above method, where the density function D is removed in a rather subtle way, the following methods solve for D and φ *simultaneously*. The price, however, is that the spherical symmetry $D = D(r)$ must be assumed, which makes sense only for pencil-beam samples.

A simple maximum-likelihood method to obtain a handle on D is that of Choloniewski (1985), who considers the probability of a galaxy lying in the interval $dM\,dm$, which is determined by $\varphi(M)$, $f(m)$, and $D(\mu)$, with μ being the distance modulus $(m-M)$. D is modeled by a steplike function, φ by a Schechter function, and the incompleteness $f(m)$ again by a Fermi-Dirac-like equation, whose best-fitting parameters are found as before by maximizing the product of the probability for the individual data points. Yet another maximum-likelihood method is that of Marshall et al. (1983), developed for quasars, and of Choloniewski (1986). The basic feature here is to treat the number of galaxies in the interval $dM\,dr$ (or of quasars in $dM\,dz$) as the result of a random process described by a Poissonian probability distribution, which has $\varphi(M)$ and $D(r)$ as ingredients. Marshall et al. (1983) have modeled φ and D by parametric expressions and determined the most likely values of the parameters in the normal way. Choloniewski (1986), on the other hand, has binned the data in the (M, μ) plane into equal intervals, which leads to steplike functions for φ and D. In the sense that no specific form of φ is assumed, his method can be called nonparametric; however, the steps could also be viewed as a set of parameters. A similar, but more general and mathematically more sophisticated, maximum-likelihood method to derive a nonparametric $\varphi(M)$ has been developed by Nicoll & Segal (1983).

3.2.4 THE C-METHOD An assumption-free method to find $\varphi(M)$ was known long before it was realized that the classical method should be replaced. This is the so-called C-method of Lynden-Bell (1971), devised and used for quasars (Jackson 1974) and only recently revived and further developed by Choloniewski (1987), who proposed its application to galaxies. The method is simple and elegant. The basic idea is to represent $\varphi(M)$ and $D(\mu)$ (assuming spherical symmetry) by superpositions of weighted δ-functions

$$\varphi(M) = \sum_i \varphi_i \delta(M - M_i) \qquad 16.$$

and

$$D(\mu) = \sum_i D_i \delta(\mu - \mu_i), \qquad 17.$$

where i denotes an individual galaxy. The problem is then to determine the coefficients φ_i and D_i. This can be achieved in an almost geometrical way by calculating for every data point M_i the quantity

$$C_i = C^-(M_i), \qquad 18.$$

which is the number of galaxies inside the region

$$M_{\min} \leqq M < M_i; \quad \mu_{\min} \leqq \mu \leqq m_{\lim} - M_i, \qquad 19.$$

where $M_{\min}$ and $\mu_{\min}$ are appropriate lower limits of M and μ. C^- is called the C-function. If the data points are ordered in such a way that $M_{i+1} \geqq M_i$, it can be shown that a very simple recursion formula holds for the coefficients φ_i (see Choloniewski 1987):

$$\varphi_{i+1} = \varphi_i \frac{C_i + 1}{C_{i+1}}. \qquad 20.$$

The analogue holds for the density coefficients D_i. Inserting the resulting φ_i and D_i into Equations 16 and 17 gives $\varphi(M)$ and $D(\mu)$, which, however, as weighted sums over δ-functions have to be smoothed (e.g. by averaging inside appropriate intervals). The revised version by Choloniewski (1987) of this method has yet to be applied to galaxies.

3.2.5 GROUPS Groups of galaxies comprise at least 70% of all galaxies in the field outside of clusters if galaxies are counted to a faint brightness limit (Holmberg 1969, Tammann & Kraan 1978). Truly "isolated" galaxies are rare (Vettolani et al. 1986). The LF of field galaxies can therefore also be approached by constructing a composite LF of groups of galaxies assuming in first approximation that field and group galaxies have identical

LFs. This method is especially valuable for the study of the *faint end* of the LF because nearby groups (notably the Local Group, and the M81 and M101 groups) have been surveyed to faint flux limits. The LF of an individual group (not the Local Group) follows, like a cluster LF, directly from the distribution of apparent magnitudes and from allowance for the distance modulus of the group. Because the individual groups possess only a few members, their LFs are usually combined into a composite group LF. As with clusters of galaxies, the difficulty lies in the identification of physical group members. Holmberg (1969), who pioneered the method, looked for faint companions close to bright spirals, which he assumed to be at the same distance. After a (very uncertain!) statistical correction for background galaxies (by counting galaxies in nearby comparison fields; see also Section 3.1), he constructed a field LF to a very faint magnitude limit of $M \sim -11$. Turner & Gott (1976b) derived a composite LF for groups that had been defined by a simple surface density criterion (Turner & Gott 1976a), without any correction for background contamination. The most reliable group LFs are based on nearby groups, where the members can be identified by morphology and velocity. A useful data base for this task is the catalog of Kraan-Korteweg & Tammann (1979), which lists all galaxies known with $v \leqq 500$ km s^{-1}. The catalog has been used by Tammann & Kraan (1978), and in a revised version by Tammann (1986), Binggeli (1987), and in the present review (Section 5, Figure 1).

An interesting variation and generalization of the group method, based on the general clustering property of galaxies, has been developed by Yee & Green (1984, 1987) and Phillips & Shanks (1987). The clustering of galaxies, as described by the correlation function (cf. Peebles 1980), means that there is (on average) an excess of galaxies on the sky around any given galaxy, which at small separations must be due to those galaxies that are physically associated with the "center" galaxy and therefore lie at the same distance. Even though it is not known individually which galaxies make up the excess, one can statistically determine the numbers of associated galaxies as a function of magnitude. If the distance of the center galaxy is known, this can be translated into a LF (Yee & Green 1984). By repeating this process for many center galaxies, an LF with good statistical accuracy at the faint end can be obtained (Phillips & Shanks 1987). Yee & Green (1984, 1987) used quasars as center "galaxies" to derive coarse galaxian LFs at high redshifts, but (as previously mentioned) the absolute magnitude calibration depends not only on the value of H_0 adopted but also on q_0 (because the redshifts are large).

Which of the many available methods to determine the LF of field galaxies should best be applied? This is difficult to answer because all postclassical methods (except the group method) have been applied to different samples of galaxies (each chosen by the developers of the method, i.e. there is yet no overlap of different studies applied to the same sample), or else have not been applied to galaxies at all. However, the distinction between parametric or nonparametric LFs is probably not essential. Working parametrically (which applies only to certain maximum-likelihood methods) has the advantage that no data binning is required, but it has the disadvantage that φ is assumed to have a certain form before the LF is determined. (The goodness of the assumption as to form can of course be tested afterward.) The opposite holds for the nonparametric case, where φ is represented by a histogram (which, however, in the end is usually fitted to a parametric expression anyway!). The expression adopted by all workers in the field is the Schechter (1976) function (Equation 13), which does model cluster and field LFs quite well (Section 4). The fitting can be done by a simple minimum-χ^2 technique, or better (to account for the large errors at the bright and faint ends) by a method that involves the so-called Eddington correction (Trumpler & Weaver 1953, Kiang 1961, Schechter 1976, Felten 1985).

More important is the role of the density function D. Strictly, the φ/Φ method and the method of Sandage et al. (1979) are the only ones that make no assumption about the form of D. This means that all information about D is lost because D must be determined independently *after* φ. All other methods supply D and φ simultaneously. In the classical method this is the mean density of the Universe, or the "normalization" of φ, which, as we emphasized earlier, has led to the unfortunate melding together of φ and D (see Section 2); otherwise it is the density as a function of the distance $D = D(r)$. The price is, of course, that an assumption has to be made about D. We know that $D \neq$ constant, i.e. the classical method is no longer viable. As stated before, methods that assume $D = D(r)$ are ideal for pencil-beam samples, which subtend small solid angles of sky, but for all-sky samples one should rather use the φ/Φ method or the method of Sandage et al. (1979), which do not assume spherical symmetry.

4. THE LUMINOSITY FUNCTION OVER ALL HUBBLE TYPES

4.1 *Cluster Galaxies*

As mentioned in Section 1.3, Zwicky (1942) postulated the exponential nature of the LF. Although he had derived the result as a theoretical necessity of thermodynamics rather than having found it observationally, he later claimed observational confirmation from cluster data (e.g. Zwicky 1957).

The completion of the Palomar Observatory Sky Survey in the late 1950s initiated a host of cluster LF studies with Abell and his collaborators as the driving force (for reviews, see Abell, 1962, 1972, 1975). The basic hypothesis was that clusters of galaxies obey a *universal* LF, which, in its integrated form $\Phi(M)$, can be modeled by two straight lines intersecting at a characteristic magnitude M^* (sometimes referred to as the "knee"). Although Abell's shape of the LF is no longer used, it is clearly a first approximation to the Schechter (1976) function in its asymptotic behavior at both the bright and faint ends.

The modern, post-Abell cluster research began with Oemler (1974), who used large-scale plates with good morphological resolution and discussed a model-free LF. His work was followed up by many authors; a partial list in Table 3 sets out those rich clusters whose LFs have been determined since 1974. The quality and depth of these investigations vary widely. Some authors give only a crude LF for a distant cluster (e.g. Iannicola et al. 1987), while others achieve great morphological detail and/or drive for very faint (surface) magnitudes [e.g. Sandage et al. (1985) in the nearby Virgo cluster]. Most cluster studies have adopted the Schechter fitting law. An exception is the "Oxford group" (Austin & Peach 1974, Austin et al. 1975, Godwin & Peach 1977, Bucknell et al. 1979, Carter & Godwin 1979, Carter 1980, Godwin & Peach 1982), who have used model-free as well as Abell fits for their LFs.

A discussion of the individual cluster LFs is not given here, nor is a detailed intercomparison of the results. This has already been provided by Dressler (1984) in his review on the evolution of cluster galaxies. Instead, we concentrate on the investigations that covered several clusters in a search for *systematic* trends.

First indications for the nonuniformity of the cluster LFs are due to Oemler (1974), who classified clusters according to galaxy content as being "spiral-rich," "spiral-poor," and "cD." The mean LFs for these three classes, which correlate with the kinematic properties of the clusters, appeared to be marginally, but significantly, different. However, Schechter (1976) fitted his expression to Oemler's clusters and, following Abell, found a rather high degree of uniformity with respect to the parameters α and M^*. This came as a surprise because significant differences could be expected in view of the cluster types presumably being in different evolutionary stages. If so, the result had to be interpreted in the sense that cluster evolution had little effect on the LF. From the near-agreement of the parameters $\alpha \sim -1.25$ and $M^*_{B_T} \sim -21.0$ for field and cluster galaxies, Schechter further concluded that the shape of the "general" LF is universal. Yet the processes of tidal stripping and dynamical friction are bound to have some bearing on the LF (cf. Dressler 1984). Dressler (1978) consequently

Table 3 Rich clusters of galaxies with LFs determined since 1974

Cluster	Investigators	Cluster	Investigators
Virgo	Bucknell et al. 1979	A1413	Dressler 1978
	Kraan-Korteweg 1981	A1553	Bucknell et al. 1979
	Sandage et al. 1985		Yamagata et al. 1984
Fornax	Jones & Jones 1980	A1656	Oemler 1974
	Caldwell 1987	(Coma)	Godwin & Peach 1977
A85	Murphey 1984		Thomson & Gregory 1980
A98	Dressler 1978		Beckman 1982
A154	Dressler 1978		Lugger 1986
A168	Dressler 1978	A1904	Oemler 1974
	Oegerle et al. 1986	A1930	Austin et al. 1975
A194	Oemler 1974	A1940	Dressler 1978
A274	Dressler 1978	A2029	Dressler 1978
A400	Oemler 1974	A2065	Bucknell et al. 1979
A401	Dressler 1978	(Corona Borealis)	
A426	Bucknell et al. 1979	A2147	Bucknell et al. 1979
(Perseus)	Egikyan et al 1985		Lugger 1986
A539	Oemler 1974	A2151	Oemler 1974
A569	Lugger 1986	(Hercules)	Bucknell et al. 1979
A665	Oemler 1974		Lugger 1986
	Dressler 1978	A2175	Oegerle et al. 1987
A744	Kurtz et al. 1985	A2197	Oemler 1974
A777	Iannicola et al. 1987		Lugger 1986
A779	Lugger 1986	A2199	Oemler 1974
A910	Iannicola et al. 1987		Bucknell et al. 1979
A999	Chapman et al. 1987		Lugger 1986
A1016	Chapman et al. 1987	A2218	Dressler 1978
A1146	Carter & Godwin 1979	A2256	Dressler 1978
A1228	Oemler 1974		Oegerle et al. 1987
A1314	Oemler 1974	A2384	Oegerle et al. 1987
A1367	Oemler 1974	A2634	Lugger 1986
	Godwin & Peach 1982	A2670	Oemler 1974
	Lugger 1986		Dressler 1978
A1377	Bucknell et al. 1979		Bucknell et al. 1979
(Ursa Major I)		0004.8−3450	Carter 1980
A1413	Austin & Peach 1974	CA 0340−538	Quintana & Havlen 1979
	Oemler 1974	Zw 1545.1+2104	Oemler 1974

searched for deviations from the first-order universality in 12 rich clusters and did indeed find significant differences. Several clusters showed an unusually flat faint end ($\alpha \sim -1$). Furthermore, the data for cD clusters supported Oemler's (1974) observations of a steeper bright end than for non-cD clusters, which was also suggested by Bucknell et al. (1979) and many others (as reviewed by Dressler 1984). The latter result could be interpreted as an evolutionary effect where the central cD galaxy formed at the expense of the next brightest cluster galaxies (Miller 1983, Malumuth & Richstone 1984, Dressler 1984). The question of whether the first-ranked cluster galaxy is within the statistics of the cluster LF or whether it is a singular object has been long debated and may now have been settled in favor of the latter possibility (cf. Dressler 1984, and references therein) for cD clusters. The jury is still out, however, for non-cD clusters [see Sandage (1988) in this volume for a review of the continuing debate].

Besides the cD effect, it has remained unclear which evolutionary processes are actually responsible for the significant variance of α and M^* among Dressler's (1978) clusters, although theoretical explanations have been suggested (Dressler 1984, Kashlinsky 1987). Lugger (1986) could not correlate the LFs of nine Abell clusters with the cluster morphology. Merritt (1984, 1985) argued on theoretical grounds that the cluster LF was determined very early during the violent relaxation phase, and that correspondingly no dependence of the LF on the present-day evolutionary stage of the clusters should be expected.

An intriguing explanation of the variance of α and M^* among clusters has been offered by Thompson & Gregory (1980). Using large-scale plates of the Coma cluster, they established the LFs of E, S0, and S + Im galaxies separately, from which they suggested that the LFs, which are clearly different for different types, remained the same in every cluster, but that the *total LF* as the sum over different types *varies according to the type mixture*. By synthesizing clusters of different type composition they were able to reproduce the variance of α and M^* observed by Dressler (1978). This hypothesis requires additional tests from many clusters with detailed morphological information that is presently only available for Virgo (Sandage et al. 1985) and Fornax (Sandage & Ferguson 1988). It should be noted that modern, sophisticated LF studies like those of Lugger (1986), Oegerle et al. (1986, 1987), and others are unsuitable for this purpose because they are based on small-scale plates where morphological binning gives unsatisfactory results. Thompson & Gregory's hypothesis is discussed and supported further in Section 6.

The limited data on the variance of the cluster LF can be explained satisfactorily as the effect of the first-ranked galaxies and the different type mixture. However, if the influence of the type mixture is denied, the remaining differences of the LF probably must be explained by evolution. Since this is not our preferred solution now, the reader is referred to the earlier review by Dressler (1984) for the consequences of this possibility.

The Schechter parameters are not listed in Table 2 because the results from individual investigations are too inhomogeneous. Different magnitude systems are used, as well as different fitting procedures. In some cases α *and* M^* were solved for, whereas in others one of the Schechter parameters was fixed a priori. A set of parameters that could be compared was provided by Lugger (1986) for nine clusters. Fitting simultaneously for α, M^*, and the normalization, she found for α a *mean* value of $\alpha = -1.24 \pm 0.22$ with the brightest galaxy excluded, or $\alpha = -1.47 \pm 0.19$ if it was included. As noted earlier by Schechter (1976), one obtains better fits if first-ranked galaxies (or cD clusters) are excluded. Therefore, Lugger's first-mentioned result is to be preferred. It is in good agreement with Schechter's canonical average of $\alpha = -1.25$, which still describes the faint-end slope of *cluster* LFs fairly well. Yet, as mentioned before, there is no single value of α that applies to all clusters (Dressler 1978). Moreover, field galaxies deviate significantly from this average, having an α closer to -1.0 (cf. Section 4.2).

It should be stressed that what most cluster studies refer to as the "faint end" lies still at fairly high luminosities, i.e. $M_{B_T} \sim -20$ or -18 at best. The exponential for the faint end defined in this way is usually a mere extrapolation of the Schechter function fitted to bright galaxies. It was therefore quite surprising that the Virgo cluster LF, which was measured to a much fainter completeness limit of $M_{B_T} \sim -14$, has confirmed a faint-end slope of $\alpha \sim -1.25$ (SBT). A somewhat shallower slope of $\alpha \sim -1.14$ has been found in a preliminary study of the Fornax cluster (Caldwell 1987). This study, however, used small-plate scale material and may suffer incompleteness.

The range of M^* for different clusters is given by Lugger (1986; see also Dressler 1978) as $M^*_{B_T} \sim -21.0 \pm 0.7$ (standard deviation) if cDs are excluded, or $M^*_{B_T} \sim -21.8 \pm 0.6$ if cDs are included. The former value is also recovered in the field (Section 4.2). The quoted magnitudes are reduced to $H_0 = 50$ throughout this paper.

In addition to the variations in α and M^*, there are clusters that seem to deny any choice of Schechter parameters (Dressler 1978). Although the significance of this effect is marginal because of the limited size of cluster samples, it is not surprising. As mentioned before, different Hubble types have quite different LFs. Depending on the type mixture, they enter into the total LF with different weight. It would be sheer coincidence if the latter could always be represented by a simple analytical formula. The *expected* ministructure of the LF over all types is discussed further in Section 5 (cf. Figure 1).

4.2 *Field Galaxies*

Felten (1977) has reviewed the classical determinations of the field LF by Kiang (1961), van den Bergh (1961), Holmberg (1969), Arakelyan & Kalloglyan (1970), Shapiro (1971), Huchra & Sargent (1973), Christensen

(1975), Turner & Gott (1976b), and Schechter (1976). He concluded that the available data could be fitted reasonably well with a Schechter function with $\alpha = -1.25$ and $M^*_{B_T} = -21.0$. The exception is the LF of groups as set out by Turner & Gott (1976b), who required a faint-end slope of $\alpha \sim -1$. This is interesting because all later determinations of the field LF point to a similarly shallow faint-end slope. Although this is a very important question concerning the frequency of true dwarfs in the field, too much weight should not be given to Turner & Gott's conclusion because their group data do not constitute a well-defined sample. Similarly, the unusual LFs found by White & Valdes (1980) for binaries and Heiligman & Turner (1980) for *compact* groups are suspected to be caused by the sample selection. In our opinion this crucial question of the faint-end slope of the LF for groups is yet to be solved.

Felten's (1977) review can be considered to be a demarcation line between eras. Two subsequent developments initiated the postclassical period of the field LF studies. First, new samples were defined. In addition to the *Reference Catalog of Bright Galaxies* (de Vaucouleurs & de Vaucouleurs 1964) and later the RC2 (de Vaucouleurs et al. 1976), the magnitude-limited all-sky redshift sample of the RSA also became available, as did the deep sample of the Harvard Center for Astrophysics with a wide sky coverage (the CfA survey of Huchra et al. 1983). Furthermore, two deep pencil-beam redshift samples were completed by Kirshner et al. (1978; the KOS sample) and by Ellis (1983) and Peterson et al. (1986; known as the Durham/AAT sample). Second, by uncovering the inhomogeneous spatial distribution of galaxies, these new redshift surveys led to the refined methods described in Section 3.

Those Schechter parameters of the *field* LF that are independent of the assumption of constant space density are set out in Table 4. To reduce the various results to the B_T system, the transformations of Felten (1985) have been adopted. As seen from Table 4, the post-1977 investigations, with the exception of that of Kirshner et al. (1983), lead consistently to a flatter faint-end slope with $\alpha \sim -1.0$ as opposed to $\alpha \sim 1.25$ for clusters (Section 4.1). As shown below, this difference of the dwarf-to-giant ratio is a natural consequence of the different type mixture in clusters and in the field. On the other hand, $M^*_{B_T}$ remains stable at ~ -21. The previously steeper slopes may be caused by the local density enhancement. Yet Felten (1985) argues that the flat slope obtained by Davis & Huchra (1982) is due to the admittedly imprecise Zwicky magnitudes. Nevertheless, the evidence is mounting for $\alpha \sim -1.0$ in the field, which would be proof that *the ratio of dwarfs to giants depends on the environmental density*. This, if true, would be a fundamental result, but again we believe the jury is still out on this matter.

Table 4 Recent work on the LF of field galaxies: best-fitting Schechter (1976) parameters (reduced to $H_0 = 50$ and the B_T-magnitude system)

Investigators	Sample[a]	m_{lim} ~	M_{lim} ~	Method (cf. Table 2)	Schechter parameter M^*	Schechter parameter α
Felton 1977 (Schechter 1976)	(Review of 9 LF studies)			I	−21.0	−1.25
Kirshner et al. 1979	KOS/RC2	15/13	−18	III	−21.0	−1.10
Kirshner et al. 1983	KOS deep subsample	18	−18	III	−21.5	−1.25
Tammann et al. 1980	RSA	(12.5)[b]	−18	V	−20.7	−1.03
Davis & Huchra 1982	CfA (north)	14.2	−16	IV	−21.0	−0.9
Choloniewski 1986				IX	−21.0	−1.09
Ellis 1983	Durham/AAT	17	−19	I	−20.7	−1.0
Shanks et al. 1984				I	−21.1	−1.0
Phillips & Shanks 1987	deep UK Schmidt	20.5	−16	XII		

[a] KOS = Kirshner et al. (1978); RC2 = *Second Reference Catalog of Bright Galaxies* (de Vaucouleurs et al. 1976); RSA = *Revised Shapley-Ames Catalog* (Sandage & Tammann 1981); CfA = Center for Astrophysics redshift survey (Huchra et al. 1983) of Zwicky galaxies (Zwicky et al. 1961–68); Durham/AAT = Durham/Anglo-Australian Telescope redshift survey (Ellis 1983, Peterson et al. 1986).

[b] No sharp magnitude limit (incompleteness function used).

4.3 *The Question of Universality*

A considerable part of our discussion of galaxian LFs has been focused on the question of whether a universal shape for the LF exists over all magnitudes, independent of Hubble type and environment. Because most cluster studies have not had sufficient type information, the question, until now, has been reduced to comparing LFs over all types in clusters with those in the field. And because the LFs have been mostly modeled by a Schechter function, the problem was further reduced to whether there is a fixed set of parameters α and M^* in all environments. Although Oemler (1974), Dressler (1978), and others have shown that there is not a unique cluster LF, the evidence was undervalued because the *mean* cluster LF seemed so similar to that of the field. From this it was argued that any real differences due to evolution had to be much larger than observed. Until recently the question of universality was therefore emphatically answered in the positive (Felten 1985).

It is now possible to investigate the LF with sufficient morphological type resolution to show that there *cannot* be a universal LF because every type has a specific LF, varying in form from nearly Gaussian to exponential. Hence, the summed LF *must* depend on the type mixture and consequently on the environment. But once this Pandora's box is opened, the question then becomes, Is the LF of a *given type* the same in clusters and in the field? This question is addressed in Section 6.

5. THE LUMINOSITY FUNCTION FOR DIFFERENT HUBBLE TYPES

The first evidence that different Hubble types have different LFs was due to Holmberg (1958, 1969); he showed, in particular, that the LF of spirals does not follow a Zwicky (1942, 1957) exponential, but that it has a maximum instead. Subsequent studies of the LF that distinguished between types generally did not go to faint enough levels to see the effect. At the bright end the LFs of different types are surprisingly similar (cf. TYS), which seemingly supports the notion of a universal LF. As is shown below, the type-specific differences occur only at fainter luminosities.

Of the many LF studies already discussed in Section 4, several did in fact differentiate, if only partially, with respect to type. Following Holmberg, these are, for the field LF, Shapiro (1971), Christensen (1975), Turner & Gott (1976b), TYS, Davis & Huchra (1982), and Ellis (1983). The LFs of E and S0 galaxies in the field were determined by van den Bergh & McClure (1979) and by Choloniewsky (1985). Type-differentiated LFs in clusters were explored by the Oxford group (for references, see Section

4.1) and, in the case of the Fornax cluster, by Jones & Jones (1980) and Caldwell (1987).

However, these investigations were too limited in absolute magnitude and/or in type resolution to reveal any significant type-dependent differences. The necessary magnitude range is available only for the local field and for very nearby clusters, while the type resolution calls for large-scale plates instead of Schmidt plates. These requirements are best fulfilled in the case of field galaxies by the "10-Mpc sample" (Kraan-Korteweg & Tammann 1979; revised by Kraan-Korteweg & Binggeli 1987) and the RSA (cf. Tammann et al. 1980), and in the case of clusters by Virgo (SBT) and Coma (Thompson & Gregory 1980). The following discussion is mainly based on these latter investigations.

An overview of the LFs of different morphological types for the local field and for the Virgo cluster is shown in Figure 1. The LFs are shown as smoothed histograms of $\log \varphi(M)$ versus absolute blue magnitude M_{B_T}. The zero point of the $\log \varphi(M)$ scale is arbitrary, but the *relative* frequency of the types within the same sample (field or Virgo) is maintained.

The data for the Virgo cluster (Figure 1, bottom) are from SBT, where a somewhat different summary diagram of the smoothed LFs was chosen (their Figure 21). The completeness limit lies at $m_{B_T} \sim 18$, which with $(m-M)^\circ = 31.7$ corresponds to $M_{B_T} \sim -13.7$. Extrapolations of the LFs beyond this limit are drawn as dashed lines. The LFs for E, S0, spiral, Irr, and dE galaxies are shown as solid lines within the completeness range. The spirals are further subdivided into Sa+Sb, Sc, and Sd+Sm systems; the subtypes are shown as dotted lines. The Irr class comprises the normal and dwarf Im's and BCDs; the latter class is also separately shown as a dotted line (for illustrations of the dwarf classes, see Sandage & Binggeli 1984). Two classes of galaxies are not shown in Figure 1 so as to avoid overcrowding, namely the rare dS0s and the exponentially increasing dE/Im's. It is not clear yet whether the dE/Im's constitute a physical transition between dEs and Im's or a gray zone because of classification problems (Sandage & Binggeli 1984, SBT). The heavy line in Figure 1 represents the total over all galaxies (including the types not individually shown).

The field data (Figure 1, top) come from two sources. For types other than E+S0, the "10-Mpc sample" (Kraan-Korteweg & Tammann 1979) was used. While the LFs of this sample were previously derived (Tammann & Kraan 1978, Tammann 1986), we use here only the 121 member galaxies in groups from a revised version (Kraan-Korteweg & Binggeli 1987). The restriction to (well-studied) groups (Local Group, M81, M101, etc.) ensures a completeness limit of roughly $M_{B_T} \sim -15$. Because the sample contains only five E+S0 galaxies, their LFs are from the RSA sample as

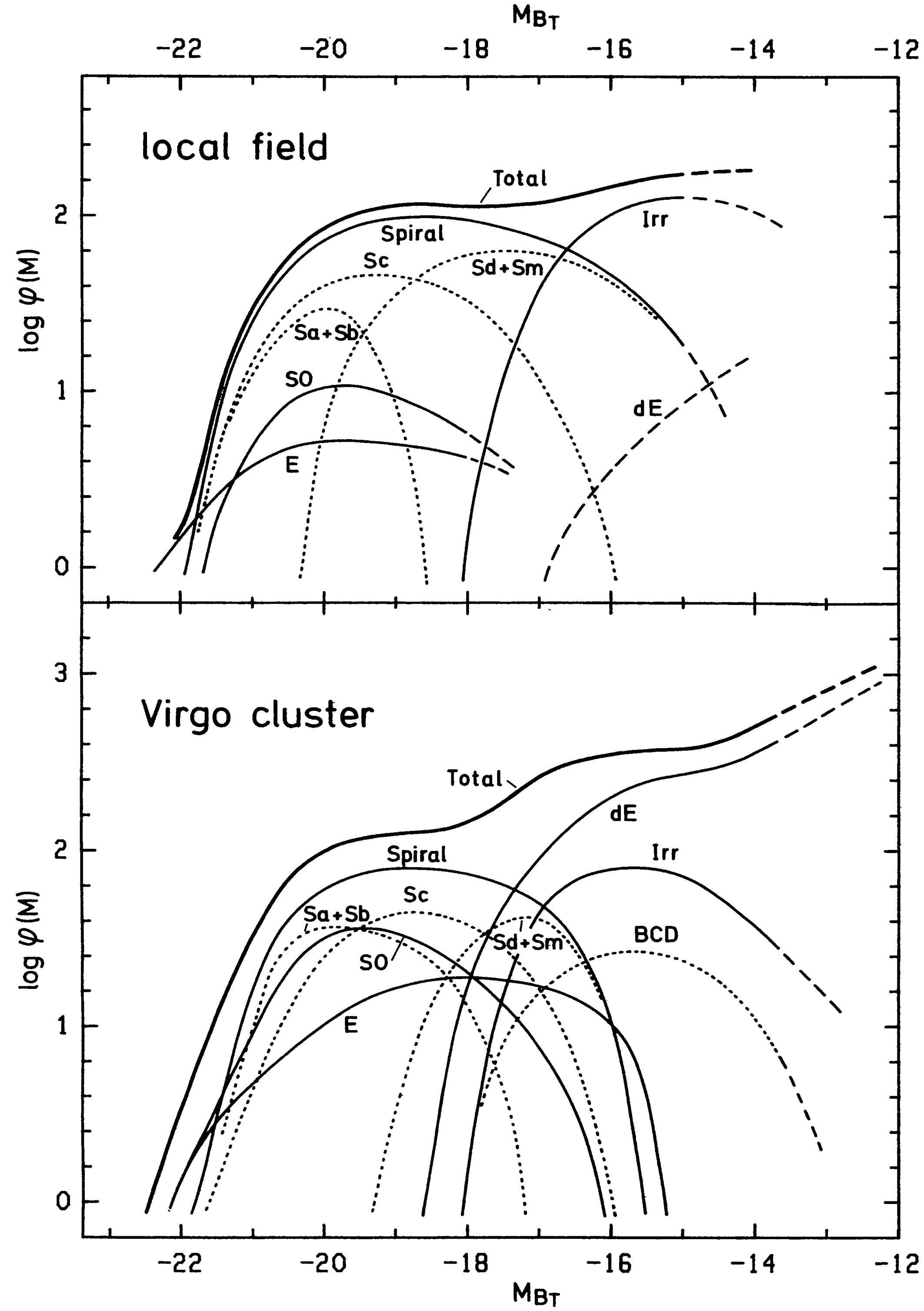

Figure 1 The LF of field galaxies (top) and Virgo cluster members (bottom). The zero point of log $\varphi(M)$ is arbitrary. The LFs for individual galaxy types are shown. Extrapolations are marked by dashed lines. In addition to the LF of all spirals, the LFs of the subtypes Sa + Sb, Sc, and Sd + Sm are also shown as dotted curves. The LF of Irr galaxies comprises the Im and BCD galaxies; in the case of the Virgo cluster, the BCDs are also shown separately. The classes dS0 and "dE or Im" are not illustrated. They are, however, included in the total LF over all types (heavy line).

determined by TYS. The completeness limit lies here at $M_{B_T} \sim -18$. The RSA LFs have been scaled such that the Es and S0s—in accord with the RSA—each contribute 5% of the total brighter than $M_{B_T} = -18$.

The type binning in Figure 1 is the same for the field as for the Virgo cluster. An exception is the BCDs, whose known number in the field sample is insignificant. Extrapolations beyond the completeness limits are again shown as dashed lines. The dE population of the field is not well known, and this type is therefore shown as a broken line. The dEs are further discussed in Section 5.2.

Comparing the LFs for the field and the Virgo cluster in Figure 1, one should remember that the Virgo data are much more reliable than those of the field. While the Virgo sample encompasses ~ 1300 member galaxies, the number of field galaxies that determine the smoothed LFs is small and drops in some cases below 10. Moreover, the Virgo galaxies could be classified from a homogeneous set of large-scale plates, whereas the types of the field sample come from various sources. Finally, the magnitudes of some of the fainter field galaxies are quite uncertain. As a consequence the Virgo cluster is taken as a basic reference in what follows; the field LFs are used only to look for *gross* deviations from the Virgo LFs. It is an important problem for the future to determine the field LF more accurately.

In the following the LFs are discussed type by type and are then summarized in Section 5.4.

5.1 *E and S0 Galaxies*

The LFs of both E and S0 galaxies have a maximum and decrease toward zero at a magnitude brighter than $M_{B_T} \sim -15$ (see Figure 1). The LF of E galaxies is skewed toward faint magnitudes, being fairly shallow at the bright end but with a sharp cutoff at the faint end. Just the opposite holds for the LF of S0 galaxies, which is skewed toward bright magnitudes, steeply rising at the bright end and shallow at the faint end. This characteristic behavior at the bright end (S0s being steeper than Es) has not only been observed in the Virgo cluster but also in the Coma cluster (Thompson & Gregory 1980) and in the field (Sandage et al. 1979, their Figure 1). As a consequence, E galaxies can reach brighter luminosities than S0s, a fact already noted by van den Bergh & McClure (1979). The brightest E galaxies in the RSA sample reach $M_{B_T} \sim -23$, while the brightest S0s reach $M_{B_T} \sim -22$ (TYS, their Figure 4; Kraan-Korteweg et al. 1984). The very brightest Es with extended envelopes are separated, following Matthews et al. (1964), into a morphologically distinct class called cD type.

The faint end of $\varphi(M)$ for S0 galaxies in the region that goes to zero has so far only been observed in the Virgo cluster (SBT). For E galaxies it has additionally been observed in the Fornax cluster (Caldwell 1987, following Wirth & Gallagher 1984). The surprisingly sharp cutoff of the LF of Es at $M_{B_T} \sim -16$ in the Virgo cluster has been confirmed by Caldwell (1987) in the Fornax cluster. However, the existence of a faint-end cutoff of the LF of Es is not without difficulties, because the Es must be separated from the dEs in the magnitude range where their LFs overlap. This range is $-18 \lesssim M_{B_T} \lesssim -16$ (see Figure 1). The problem is discussed in the next section.

5.2 *Dwarf Elliptical Galaxies*

E galaxies and dEs are morphologically distinct (Wirth & Gallagher 1984, Sandage & Binggeli 1984). Es are in general compact and of high surface brightness. Their compactness increases with decreasing luminosity (Kormendy 1980, 1985). In contrast, dEs are diffuse, i.e. they have low surface brightness. They become increasingly diffuse with decreasing luminosity (Baade 1944, Caldwell 1983, Binggeli et al. 1984, Bothun et al. 1986, Caldwell & Bothun 1987; for an illustration, see Sandage & Binggeli 1984). This dichotomy is classical: M32 *versus* NGC 205 and the Fornax and Sculptor dwarfs, as first discussed by Baade (1944). The distinction of Es and dEs, best visible in a plot of central surface brightness *versus* total magnitude (Kormendy 1985, his Figure 3) and suggested also by the different trends in an *effective* surface brightness–M_{B_T} diagram (BST, their Figure 8), must almost certainly mean that the two classes are of different origin (Kormendy 1985, Dekel & Silk 1986). This is also supported by the fact that the LFs of Virgo Es and dEs do not merge well into *one* LF. However, in the overlapping range ($-18 \lesssim M_{B_T} \lesssim -16$) Es and dEs, because of the cosmic scatter of their parameters, are sometimes hard to separate, even with the help of luminosity profiles (Caldwell & Bothun 1987). A confirmation that Es and dEs can presently be separated successfully might be seen in the fact that their LFs agree very well in the Virgo cluster (SBT) and Fornax cluster (Caldwell 1987), but the morphological separation of the latter cluster rests on plates with too small a scale to provide a stringent test.

The dwarf elliptical LF in the Virgo cluster can be modeled fairly well by a Schechter function with parameters $\alpha = -1.35$ and $M^* = -17.4$ (note the very steep increase at faint magnitudes compared with the field value of $\alpha \approx -1$). A notable deviation from a Schechter fit is the maximum at $B_T \sim 16$ ($M_{B_T} \sim -16$; cf. Figure 1, bottom). This maximum was also found in the Fornax cluster by Caldwell (1987) but not confirmed by Ferguson & Sandage (1987). In the Virgo cluster, dEs are still rising exponentially at $M_{B_T} \sim -14$ (the completeness limit) and diffuse systems are known to exist down to $M_{B_T} \sim -12$. Still fainter systems are expected because local dEs are known at roughly $M_{B_T} \sim -8$ (Binggeli et al. 1984). The work of Impey et al. (1987) can be expected to shine light on this dark matter.

A detail worth noting is that dEs come in two subclasses: with or without central semistellar nucleus [cf. Sandage & Binggeli (1984) for illustrations]. The detectability of the nuclei is correlated with absolute magnitude; the mixing ratio of nucleated to nonnucleated dEs decreases from 1 to ~ 0.1 almost linearly from $M_{B_T} \sim -18$ down to ~ -12. Taken at face value, this means that the LF of nucleated dEs has a maximum, and that only the number of nonnucleated dEs takes off exponentially (SBT). However, the classification of nucleated dEs may be affected by another iceberg effect. Very faint nuclei may have passed unnoticed. The correlation of the galaxy and nucleus luminosities makes it possible, in fact, that all dEs have some kind of nucleus.

The field LF of dEs is difficult to establish because these galaxies are rare in low-density regions. As seen in the Virgo cluster (BST), their number frequency correlates strongly with local density. In this sense they behave like E galaxies, whose frequency-density relation was established by Dressler (1980). A systematic search for field dEs is being conducted by Binggeli et al. (1988). At present the only available evidence for fairly low-density regions comes from the Local Group and the M81 group. The latter was searched for dwarfs down to $M_{B_T} \sim -12.5$ (Börngen et al. 1982, Karachentseva et al. 1985; cf. also Sargent 1986), resulting in 10 dEs with $-15 \leqq M_{B_T} \leqq -12.5$. Their distribution is compatible with an exponential increase toward fainter objects. Therefore, the dashed curve of the field dEs in Figure 1 (top) is put in schematically with the shape of the LF of the Virgo dEs, but appropriately scaled to the relative frequency of dEs in the Local Group and the M81 group.

The dwarf S0s in the Virgo cluster have a narrow Gaussian LF centered on $M_{B_T} \sim -17$ (see SBT); they are not separately shown in Figure 1.

5.3 *Spiral and Irregular Galaxies*

Long ago, Holmberg (1958, 1969) observed that the LF of spiral galaxies has a maximum. There are no dwarf spirals. The LF of all Virgo spirals can be modeled by a Gaussian with a mean of $M_{B_T} = -18.4$ and a dispersion of $\sigma_M = 1.5$ mag (SBT). The overall form of the spiral LF is not too dissimilar to that of Es and S0s, especially at the brighter end (cf. Figure 1), which of course is the root of the simplified classical view that the LF does not depend on type. The spiral subtypes have been combined in Figure 1 into the classes Sa + Sb, Sc, and Sd + Sm so as not to complicate the diagram by a finer subdivision and to avoid small number statistics. The general trend is that early-type spirals are brighter in the mean than late-type spirals. But the Hubble spiral sequence does not order the galaxies monotonically into a luminosity sequence, because the mean luminosity of Sb (and Sab) galaxies is brighter than that of Sa's (SBT) (not shown in Figure 1 because of the binning). Sb galaxies are indeed among the intrinsically brightest galaxies in the Universe (cf. RSA, p. 94).

The spiral LFs in Figure 1 show a similar behavior for the field (top) and the Virgo cluster (bottom). They have the same maxima M_{B_T} values in the cluster and the field, and further the subtypes are ordered in the same sequence. The difference shown in Figure 1 lies in the dispersion, but whether this difference is truly significant is questionable. To within the accuracy (small number statistics and inhomogeneous type determinations in the 10-Mpc sample) they are the same, and we advocate below that they are, in fact, the same.

The LFs of spirals become narrower near-Gaussians than those shown in Figure 1 when they are binned into the van den Bergh luminosity classes. Corresponding LFs for field galaxies in the RSA have been calculated by TYS, Kennicutt (1982), and Kraan-Korteweg et al. (1984).

If Im's are considered as the homologues of spirals, but with too low a mass to form spiral arms, the two classes of star-forming galaxies may be fitted by a single LF. The result is a clearly peaked bell-shaped curve of which only the faintest end is ill defined (cf., for the field, Tammann & Kraan 1978; for Virgo, by construction from data given by BST). If Im's are taken as an individual class, their LF has a maximum at rather faint magnitudes, namely at $M_{B_T} \sim -16$ for the Virgo cluster and roughly at -15 for the nearby groups, where this class dominates the faint end of the total LF (see Figure 1, but recall that the curve labeled "Irr" comprises Im's and BCDs; in the field the BCD contribution is negligible). The magnitude difference between the maxima in the Virgo cluster and the nearby groups is not significant, but the existence of the maxima seems to be well established by the data, particularly in the case of Virgo. It is therefore surprising that Scalo & Tyson (1987) have advocated recently that the Virgo maximum is an artifact because SBT have missed enough low-surface-brightness Im's owing to the iceberg effect to make the real LF an exponential. We return to this possibility below.

The LF of the Virgo BCDs is shown in Figure 1 (bottom). Because these galaxies are relatively easy to detect, their LF maximum at $M_{B_T} \sim -15.5$ is certainly real, although there remains an ambiguity as to cluster membership for some high-surface-brightness BCDs that are difficult to distinguish from background objects on purely morphological grounds. BCDs are rare in the field and in groups (Tammann 1986), and this rarity is at most only partially due to the inhomogeneous classification of our corresponding sample. In any case, no LF can be given for them.

Im's, especially the faint and low-surface-brightness ones (which are rather smooth in appearance), are often hard to distinguish from dEs. In the Virgo cluster there is a whole class of objects for which the distinction is impossible. These objects were consequently called "dE or Im." The LF of these interlopers is *exponential*, similar to that of dEs (not shown in Figure 1; cf. SBT). It is unclear whether these systems run counter to a clear classification because of insufficient resolution, or whether they constitute a genuine (physical) transition between dE and Im. In favor of the latter hypothesis is the observation that the structural parameters of Im's and dEs are very similar—for instance, both exhibit exponential radial luminosity profiles (Wirth & Gallagher 1984, Binggeli et al. 1984, Kormendy 1985, Binggeli 1986, Bothun et al. 1986). This behavior has prompted several suggestions as to the origin of dEs and of diffuse dwarfs in general (dE+Im) (Lin & Faber 1983, Kormendy 1985, Dekel & Silk 1986, Silk et al. 1987). If, on the other hand, a large fraction of the transition class were true Im's, they could account for the exponential LF of Im's as suggested by Scalo & Tyson (1987).

5.4 *Summary*

Figure 1 clearly shows that the LFs of different types are widely different, and that the total LF over all types is the sum of many single, mostly bell-shaped curves. Discussing these samples in terms of a Schechter function over all types is like covering a wealth of details with a thick blanket. There are humps and bumps in the total LF according to the underlying type-specific LFs. The double wave of the Virgo total LF (Figure 1, bottom), for instance, cannot be modeled by a Schechter function. Figure 1 also reveals clearly the difference between the faint-end slopes of the total LFs of the field and of the Virgo cluster. No evidence is found that the *type-specific* LFs are different in the field and in Virgo. But owing to the variation of the type mixture as a function of environment, *there cannot be a universal total LF.* This conclusion leads to a new approach to the problem of universality, as set out in the following section.

6. THE LUMINOSITY FUNCTION FOR DIFFERENT ENVIRONMENTS

In Section 5 it became evident that different LFs apply to different morphological types T. Hence, we have $\varphi = \varphi_T(M)$. Throughout this review we have also emphasized that the general $\varphi(M)$, summed over all types, depends on the environmental density. This is especially true for the faint-end slope, which is found to be notably different for field and clusters, varying among the clusters (cf. Section 4) as well as between clusters and the field.

It is clear that we must distinguish between an environmental dependence of the general $\varphi(M)$, which is merely due to the type mixture differences, and a dependence of the type-specific $\varphi_T(M)$'s on the environment. As previously mentioned, we hypothesize that there is only a negligible environmental variance of the type-specific $\varphi_T(M)$'s, and therefore that the variation of the type mix is the major parameter responsible for the variance of the total $\varphi(M)$ from sample to sample.

But before we come to this simple scheme in Section 6.3, it is desirable to approach the problem in a more systematic and general way. We consider first giving up the artificial dichotomy of "clusters" and "field" and substituting for it the notion of a continuous variation of density D. The distribution of galaxies is known to be inhomogeneous on all scales up to at least 100 $(50/H_0)$ Mpc. A rich cluster of galaxies is like a Matterhorn in a grand Alpine landscape of mountain ridges and valleys of lengths up to 100 km. Correspondingly, the galaxy density $d(x, y, z)$ is a continuous parameter, and the most general approach is therefore to treat the specific $\varphi_T(M)$ as a continuous function of this density, $\varphi_T(M, D)$. The consequence is that the universally adopted assumption of separability of φ and D, as expressed in Equation 8, should now be abandoned, giving way to a general LF-density relation through the Dressler (1980) density-type relation.

6.1 *The Morphology-Density Relation*

Following earlier work discussed in Section 1 (cf. also Hubble & Humason 1931, Morgan & Lesh 1965, Abell 1965, Oemler 1974, Melnick & Sargent 1977) Dressler (1980) discovered that the (projected) local density of galaxies governs the mixture of Hubble types in any local environment of a cluster, independent of the cluster's global environment. Subsequent work by Bhavsar (1981), de Souza et al. (1982), and Postman & Geller (1984) has extended Dressler's relation to groups and the general field.

But the Dressler effect holds not only for the main Hubble types but also for spiral subclasses as well, as convincingly shown by Giovanelli et al. (1986) in their study of the Pisces-Perseus supercluster and anticipated by Gisler (1980). [Barred spirals, strangely, may be an exception (Kumai et al. 1986).] A strong morphology-density relation has also been found for the dwarf galaxies (dE vs. Im) in the Virgo cluster (Binggeli et al. 1988). *The whole Hubble sequence seems affected, down to very fine morphological detail.*

As the shapes of the LFs for the different types vary little (or, by our working hypothesis, not at all) for the two different environments, which now represent merely two different mean densities (low and high) in the continuous range of D, one can estimate the relative frequency of a given type by reading the amplitude at the relevant magnitude M_{B_T}. It is clearly visible from Figure 1 not only how strongly the early types (E, S0, dE) are suppressed in the field, but also how much the late types (Sa+Sb, Sc, Sd+Sm, Irr) are boosted—the later the type, the larger the boost. (Note the logarithmic scale of Figure 1, which conceals the importance of the effect.)

6.2 *Toward a Luminosity Function–Density Relation*

It is easy in principle to calculate the local density in any environment and to construct and intercompare $\varphi_T(M)$ for different density bins. The problem is that the binning must, by necessity, be very coarse because with finer binnings the galaxy numbers would be too small to give any weight to the resulting $\varphi_T(M)$. The coarsest binning is the traditional "field"-"cluster" separation emphasized throughout this review. But the LFs for *different types* have usually not been compared for the two environments, or at most only at the bright end, where every LF is approximately the same (cf. Sections 4, 5). The treatment of a cluster as a mean environment is quite unsatisfactory because the local (projected) density in a cluster can vary by a factor of 100 (Dressler 1980); indeed, the morphology-density relation applies even to different regions within a given cluster (which also proves beyond doubt either that virial mixing has not occurred *or* that galaxies have changed morphological type within a cluster!).

Godwin & Peach (1977) were the first to look for differences of the LF *within* a cluster. Constructing $\varphi(M)$ for the Coma cluster separately for an inner and an outer zone, they found a significant deficiency of bright galaxies in the inner part (neglecting the two dominant cD galaxies). The same phenomenon for the inner and outer regions of two other clusters has recently been reported by Lugger (1987). Probably this is simply a refinement of the observation of Oemler (1974) and Dressler (1978) that cD clusters on the whole have a deficit of bright galaxies, perhaps to be interpreted as the result of merging and tidal stripping (Dressler 1984; see also Section 4.1).

Dressler (1980) has compared the LFs of S+Irr and S0 galaxies in low-density and high-density regions of his 55 clusters and found no evidence for any difference. B. Binggeli (unpublished work) did the same for the Virgo cluster alone (which is feasible because of the large number of known members), separately for E, S0, dE, spiral, and Irr galaxies, and again found no clear indication that the *type-specific* cluster LFs depend on the local density. There is only a hint for E and S0 galaxies to be brighter in higher density regions.

The topic of the LF-morphology-density relation for field galaxies has been addressed by de Souza et al. (1985), Choloniewski & Panek (1987), and Einasto & Einasto (1987). All three studies are essentially based on the CfA redshift survey (Huchra et al. 1983). In the first two studies the local spatial densities have been estimated by a nearest-neighbor algorithm. No density dependence of the LFs was found for the main Hubble types. (A marginal effect was reported for S0s.) Instead of constructing full LFs, Einasto & Einasto (1987) have chosen to calculate the mean absolute magnitude of the three brightest galaxies for different morphological types and environments. Evidence was found that the brightest galaxies in groups and clusters are brighter than those in the field by up to 1 mag; this effect is claimed not to be caused by the Malmquist bias.

6.3 *A Case for Simplicity*

Apart from this last-mentioned suggestion that the *brightest* galaxies lie preferentially in dense environments (probably as a result either of stochastic sampling or of dynamical evolution), there is no evidence that the shape of the *type*-specific LFs, $\varphi_T(M)$, depends on the local density (see Section 5 and Figure 1). Perhaps the strongest case against a density dependence of $\varphi_T(M)$ comes from the Virgo and Coma clusters. Despite the very different type mixtures of these clusters (namely E : S0 : S+Irr = 12 : 26 : 62% for Virgo, but 44 : 49 : 7% for Coma), the LFs of these three bins of galaxies are essentially identical (Binggeli 1987).

Adopting the hypothesis that φ_T is independent of D, i.e. $\varphi_T(M, D) = \varphi_T(M)$, one can treat the total $\varphi(M)$ as a function of type and density by expanding the formalism in Section 2 (Equations 8–11) as

$$\nu_T(M, x, y, z) = \varphi_T(M, D) \cdot D_T(x, y, z) = \varphi_T(M) \cdot D_T(x, y, z), \qquad 21.$$

where

$$\int_{-\infty}^{+\infty} \varphi_T(M)\, dM = 1. \qquad 22.$$

This means that the separability of φ and D (Equation 8) is recovered for any given T.

If $f_T(D)$ is the relative frequency of a given type over all magnitudes as a function of the density, one can write

$$D_T(x, y, z) = f_T(D) \cdot D(x, y, z), \qquad 23.$$

where

$$\sum_T f_T(D) = 1. \qquad 24.$$

From Equations 21 and 23, the sum over all types becomes

$$\nu(M, x, y, z) = \sum_T \varphi_T(M) \cdot f_T(D) \cdot D(x, y, z), \qquad 25.$$

and the LF over all types is therefore

$$\varphi(M, D) = \sum_T \varphi_T(M) \cdot f_T(D), \qquad 26.$$

where (from Equations 22 and 24),

$$\int_{-\infty}^{+\infty} \varphi(M, D)\, dM = 1. \qquad 27.$$

Equation 26 clearly reflects that the general LF is *not universal.* Instead, it is the sum over all type-specific LFs weighted by the density-dependent type fraction.

The difficulty with the normalization in Equations 22 and 27 is that the integration of φ to infinity is impractical because at least the LF of dEs is exponential. A solution to the problem was suggested with Equation 11, where φ was integrated to a specified absolute magnitude $\tilde{M}$. If this procedure is again adopted here, the type fraction $f_T(D)$ also becomes a function of $\tilde{M}$. Therefore, as stressed already by de Souza et al. (1985), the *same magnitude cutoff* $\tilde{M}$ *must be used for the morphology density relation.* Equation 26 can then be reformulated as

$$\varphi(M, D) = \sum_T \varphi_T(M) f_T(D, \tilde{M}), \qquad 28.$$

where

$$\int_{-\infty}^{\tilde{M}} \varphi_{\mathrm{T}}(M)\, dM = 1 \qquad 29.$$

and

$$\sum_T f_T(D, \tilde{M}) = 1. \qquad 30.$$

A remaining question is to what detail the type binning should be carried. The minimum detail required for spirals is the differentiation into subtypes, because the subtypes have different LFs (cf. Figure 1) and follow different morphology-density relations, as discussed in Section 5.3. Therefore, in order that φ_T be universal, the morphological binning should be *at least* as fine as that used in Figure 1. But it is not clear whether this degree of differentiation is sufficient. It was conjectured in Section 6.1 that the morphology-density relation may extend down into the finest morphological separation. However, the morphological binning must not reach the point where every galaxy is treated as an individual; otherwise the hypothesis that φ_T is universal would be trivially true. A fine T-binning is often also impractical, e.g. for the LFs of distant clusters. Nevertheless, even with a coarse binning the hypothesis of constant φ_T may be a valid and useful *approximation.*

The precise universality of φ_T is not to be expected for other reasons as well. Dynamical interaction with the intracluster medium and between galaxies in the central parts of clusters has some influence on the evolution (Dressler 1984) and also on $\varphi(M)$ (cf. Section 4.1). But, following the introduction of a general LF over all types by Abell (1962, 1964), its consolidation by Schechter (1976), and its canonization by Felten (1977, 1985), the present hypothesis that the type-specific $\varphi_T(M)$ is universal, independent of the environment, is probably the next logical step for an understanding of the LFs of galaxies. Of course, its validity must be tested further.

7. PROBLEMS AND PROSPECTS

It would be fair to say that the principal features of the general LF (i.e. summed over Hubble types in an "average" environment) are now well known, as distilled from the studies of field and cluster samples that were reviewed in previous sections. What then remains to be established and with what means?

Once we adopt the premise that the individual type-specific $\varphi_T(M)$ functions are more fundamental than their sum in any particular environment (e.g. spiral-rich or spiral-poor), the emphasis then shifts to determining $\varphi_T(M)$ rather than $\Sigma_T\,\varphi_T(M)$. It is here that our present knowledge is least secure.

We would like to test if the shape of $\varphi_T(M)$ depends on density; it was assumed not to be so in previous sections, but only because our present knowledge is incomplete. Measurements of $\varphi_T(M)$ are relatively easy in clusters (taking note of the membership problem) because all galaxies are at the same distance. But cluster studies only touch the high-density end of any $\varphi_T(M, D)$ relation. To sample the crucial low-density regime we must find $\varphi_T(M)$ in various (and different) parts of the field, or, speaking as before, in the high and low Alpine valleys and passes away from the Matterhorns. It is this excursion deep into the field that will usher in the next era of the work.

The usual technical problems will remain, each to be solved for particular projects. These include measurements of apparent magnitudes, surface brightness, morphological types, and redshifts (or other distance indicators).

On the theoretical side, once the $\varphi(M, D)$ functions are known for each T, what do these probability distributions tell us about galaxy formation and evolution? [The simplest example of a theoretical deduction from an observed $\varphi_T(M)$ distribution is that because no dwarf spirals exist, formation of spiral arms requires that the galaxy mass (which governs the maximum rotational velocity for a disk structure) be larger than a particular value. To make such a statement requires the knowledge that $\varphi_T(M) \to 0$ as $M \to \infty$ for spiral types.]

7.1 *Observational*

7.1.1 NATURE OF RELEVANT SURVEYS A condition of utmost importance in finding $\varphi_T(M, D)$ in the field is that the galaxy sample to be studied have exquisitely defined selection criteria (i.e. complete to a given magnitude, to a given redshift, and to a given surface brightness for a given type). Since no sample is complete, the incompleteness must be known and factored into the solution for the effective volume surveyed.

7.1.2 MAGNITUDES To date, field galaxy LF studies still rely on the Zwicky magnitude system, known to have systematic errors depending on the particular catalog volume (and therefore a function of declination) (Kron & Shane 1976). The errors are also functions of angular diameter and surface brightness [and therefore of type because $SB = f(T)$] (Balkowski et al. 1974, Fisher & Tully 1975). The present methods for correcting this central and crucial data base remain *patchwork* and are unsatisfactory even at the $+0.5$ mag level, especially because the corrections are a function of magnitude.

What is needed is a new all-sky magnitude survey to at least the limit of the CfA redshift survey in the north and its companion redshift study in the south (da Costa et al. 1984). The European Southern Observatory (ESO) magnitude survey of A. Lauberts, now in progress, is a step in this direction.

All optical LFs of galaxies to date have been restricted to B (or m_{pg}) magnitudes. Luminosity functions in other wave bands are of course possible. In fact, they would be nontrivial because of the color dependence on galaxy type and luminosity. Moreover, for high-redshift clusters the effects of the K-correction could be circumvented if their LFs were observed at the rest-frame wavelength of the B system.

7.1.3 SURFACE BRIGHTNESS It was mentioned in Section 2 that the actual observational limit at the faint end is not defined by total magnitude, but rather by surface brightness. A systematic survey of low-surface-brightness objects would be extremely useful for getting a handle on the density of dwarfs in the local field and for further checking the possible existence of bright, low-surface-brightness galaxies (Disney 1976, Disney & Phillips 1983) that may threaten the $\varphi(M)$'s. The new IIIaJ Palomar Schmidt survey appears promising in this respect (cf. Sargent 1986). A definitive solution to the problem of hidden objects of extremely low surface brightness ($\mu > 26$ B mag arcsec^{-2}) is expected from D. Malin's technique (Impey et al. 1987).

The special study of low-*SB* galaxies is particularly important to solve the problem of the giant E/dwarf dE ratio as a function of density. Do dE dwarfs only occur as satellites of giant galaxies or in clusters, or do they occur isolated in the field? This question can be answered by testing if the α-parameter of the Schechter function fainter than, say, $M_{B_T} \sim -17$ has the same value in the field (or even in very low-density regions of clusters) as in the cluster centers. The problem is central to the question of biased galaxy formation. It is equivalent to asking if the relative number of dwarfs to giants is larger in voids than in regions replete with giants. The survey of Binggeli et al. (1988), mentioned in Section 1.2.3, aims at answering this question.

A second problem concerning the $\varphi(M)$ for Im galaxies is also in need of solution. From both the 500 km s^{-1} (distance-limited) Kraan-Korteweg & Tammann (1979) sample and the Virgo cluster catalog (BST), the Im LF was found to show a maximum at $M_{B_T} \sim -15.5$, from which it decreases to zero. It would be an important result if all currently star-producing galaxies show such a maximum rather than an exponential increase toward fainter luminosities. However, Scalo & Tyson (1987) suggest that we have missed the faint *SB* Im's in our Virgo cluster survey (BST) and that the relevant $\varphi(M)$ does increase exponentially, similar to $\varphi(M)$ for dE galaxies. The problem hinges on the true nature of the transient class ("dE or Im") of BST, which has indeed an exponential LF. This obviously is a central point for ideas of galaxy formation and evolution and must be tested by observation rather than hypothesis.

7.1.4 MORPHOLOGICAL RESOLUTION Once the LF is shown to depend on the morphological type, classifying galaxies becomes more important again. Most LF studies in clusters are based on Schmidt plates because they are convenient for covering the required large fields. However, such plates do not permit a fine type binning. Only the main classes, if any at all, can be distinguished for high-*SB* galaxies fainter than $B_T \sim 12$ on such small-scale plates. Large-scale plates with a wide field of view are also needed for clusters, such as the Las Campanas Observatory plates used by Dressler (1980) for his study of 55 clusters. These plates would be ideal for obtaining $\varphi_T(M)$ for many clusters (the Las Campanas 100″ telescope is in fact the only instrument suitable for the task) and for testing the hypothesis of universal $\varphi_T(M)$'s. Good, homogeneous classifications do exist for bright field galaxies (RC2, RSA) but not for the 500 km s^{-1} catalog (Kraan-Korteweg & Tammann 1979), which would otherwise be an ideal data base for the local $\varphi(M)$.

An enormous opportunity now exists to study many aspects of the field LF by combining magnitudes and morphological types with the modern redshift surveys (CfA survey, da Costa et al. 1984) that are designed to be complete to an intermediately faint Zwicky magnitude ($m_{pg} \sim 14.2$). *Complete* (all-sky) redshift surveys to fainter magnitudes are impractical, but smaller area surveys patterned after that of Kirshner et al. (1978) are in progress at several observatories. Accurate galaxy types for each of these large survey catalogs, especially for the low-*SB* late Hubble types, would enhance our knowledge by a quantum leap.

7.1.5 ENVIRONMENT The correlation of $\varphi(M)$ with the environmental galaxy density requires reliable estimates of the latter. The densities should account for the total local galaxy number to a uniform lower luminosity limit. In the field, redshift data are needed for all candidate galaxies to exclude foreground and background objects. Methods to derive the local density are described by Postman & Geller (1984) and Choloniewski & Panek (1987). It is not yet clear how big a volume around each galaxy should be considered to achieve the best correlation between density and morphology, and hence $\varphi(M)$. In clusters one can derive smoothed three-dimensional radial density profiles if a cluster model is assumed. But subclustering, which may be important, can only be approximated at best because it will remain uncertain whether a given galaxy is connected to the local structure or whether it is projected onto it.

Good control of the environment is of course needed if the universality of the type-specific LFs, $\varphi_T(M)$, is to be checked in a large number of field and cluster regions. Provided that $\varphi_T(M)$ is fixed for any given type T, it will be possible to synthesize $\varphi(M)$ summed over all types as a function of the local density by weighting sets of standard $\varphi_T(M)$'s with the type fraction appropriate for that density. The shape of the synthetic total $\varphi(M)$ can then be compared with observations in corresponding regions without further regression to morphology.

The observed total $\varphi(M)$'s may eventually turn out not to be a simple function of the local galaxy density, because this parameter alone can hardly determine the actual local type mixture. The latter must also depend on the dynamical history of a region. For instance, the present infall of galaxies, mainly of spirals, into the Virgo cluster will increase the spiral population of this cluster by $\sim$40% over the next Hubble time (Tully & Shaya 1984) and will at the same time increase the local density. This is contrary to the average density-type relation that predicts a lower spiral fraction for higher density. On the other hand, the Hydra cluster, which lies isolated in space (Richter et al. 1982), can hardly increase its present spiral population.

7.2 *Theoretical*

The role of the LF in practical cosmology is discussed in Section 1 and is not further taken up here.

It has always been clear that the LF of galaxies is fundamental for an understanding of galaxy formation. Many of the processes involved in the formation events undoubtedly have determined the LF of galaxies. However, the problem is so difficult that it is not surprising that, to date, theoretical interest has essentially been with the cosmological applications and not with the physics that determines the LF. Notable exceptions are Zwicky's (1942, 1957) attempt to explain the exponential form of the LF on the grounds of statistical mechanics, and Press & Schechter's (1974) self-similar gravitational condensation model of galaxy and cluster formation, which closely reproduces a Schechter-type LF. An approximately exponential behavior of the LF of galaxy systems (from single galaxies and small groups to rich clusters; Bahcall 1979) is generally in qualitative agreement with a hierarchical clustering scenario of galaxy formation (e.g. Silk & White 1978).

It must be suspected that the type-specific LFs, which now are established (i.e. the *differences* between the types), carry more stringent clues to the formation and evolution of galaxies than the general $\varphi(M)$. However, galaxy formation theories have just barely reached the stage of being able to address the origin of the Hubble sequence.

The most basic morphological detail of the LF may be that all classes of high-surface-brightness (i.e. "normal") galaxies have a bell-shaped LF, whereas low-surface-brightness galaxies (dEs and possibly Im's), and *only* these, have an exponential or Schechter-type LF. These two basic branches of galaxies also appear distinct in the magnitude–surface brightness diagram (Wirth & Gallagher 1984, Kormendy 1985) that prompted Dekel & Silk (1986) to develop a model in which the low-surface-brightness dwarfs are the residuals of a mass-loss instability that occurs below a certain critical mass. In this case, the present Im's may be dEs repowered by infalling cooling gas (Silk et al. 1987). This mechanism, if restricted to the bigger systems, could possibly account for the bell-shaped LF of Im's, as well as for the exponential LFs of dE's and the transition class "dE or Im." An explanation of the LF dichotomy (bell-shaped vs. exponential) has been attempted by Schaeffer & Silk (1986). A more detailed understanding of the $\varphi_T(M)$'s must await further theoretical progress.

Besides the variety of shapes of the type-specific LFs, the observation that $\varphi_T(M)$ is independent of environment, if confirmed, will also put very strong constraints on theories of galaxy formation.

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