Magnetic Fields in the Early Universe

Dario Grasso[a1] and Hector R. Rubinstein[b2]

[1] *Dipartimento di Fisica "G. Galilei", Università di Padova,*
Via Marzolo, 8, I-35131 Padova, Italy,
and
I.N.F.N. Sezione di Padova.
[2] *Department of Theoretical Physics, Uppsala University,*
Box 803, S-751 08 Uppsala, Sweden,
and
Fysikum, Stockholm University, Box 6730, 113 85 Stockholm, Sweden.

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[1]dario.grasso@pd.infn.it
[2]rub@physto.se

Abstract

This review concerns the origin and the possible effects of magnetic fields in the early Universe. We start by providing to the reader with a short overview of the current state of art of observations of cosmic magnetic fields. We then illustrate the arguments in favour of a primordial origin of magnetic fields in the galaxies and in the clusters of galaxies. We argue that the most promising way to test this hypothesis is to look for possible imprints of magnetic fields on the temperature and polarization anisotropies of the cosmic microwave background radiation (CMBR). With this purpose in mind, we provide a review of the most relevant effects of magnetic fields on the CMBR. A long chapter of this review is dedicated to particle physics inspired models which predict the generation of magnetic fields during the early Universe evolution. Although it is still unclear if any of these models can really explain the origin of galactic and intergalactic magnetic fields, we show that interesting effects may arise anyhow. Among these effects, we discuss the consequences of strong magnetic fields on the big-bang nucleosynthesis, on the masses and couplings of the matter constituents, on the electroweak phase transition, and on the baryon and lepton number violating sphaleron processes. Several intriguing common aspects, and possible interplay, of magnetogenesis and baryogenesis are also dicussed.

Contents

Introduction

Magnetic fields are pervading. Planets, stars, galaxies and clusters of galaxies have been observed that carry fields that are large and extensive. Though strong homogeneous fields are ruled out by the uniformity of the cosmic background radiation, large domains with uniform fields are possible.

A crucial ingredient for the survival of magnetic fields on astrophysical scales is for them to live in a medium with a high electrical conductivity. As we shall see in Chap.1, this condition is comfortably fulfilled for the cosmic medium during most of the evolution of the Universe. As a consequence, it is possible for magnetic fields generated during the big-bang or later to have survived until today as a relic.

To establish the existence and properties of primeval magnetic fields would be of extreme importance for cosmology. Magnetic fields may have affected a number of relevant processes which took place in the early Universe as well as the Universe geometry itself. Because of the Universe high conductivity, two important quantities are almost conserved during Universe evolution: the magnetic flux and the magnetic helicity (see Sec.1.4). As we will see, valuable information about fundamental physics which took place before the recombination time may be encoded in these quantities.

In the past years a considerable amount of work has been done about cosmic magnetic fields both on the astrophysical and the particle physics point of view. The main motivations of such wide interest are the following.

The origin of the magnetic fields observed in the galaxies and in the clusters of galaxies is unknown. This is an outstanding problem in modern cosmology and, historically, it was the first motivation to look for a primordial origin of magnetic fields. Some elaborated magnetohydrodynamical (MHD) mechanisms have been proposed to amplify very weak magnetic fields into the μG fields generally observed in galaxies (see Sec.1.1). These mechanisms, known as *galactic dynamo*, are based on the conversion of the kinetic energy of the turbulent motion of the conductive interstellar medium into magnetic energy. Today, the efficiency of such a kind of MHD engines has been put in question both by improved theoretical work and new observations of magnetic fields in high redshift galaxies (see Sec.1.2). As a consequence, the mechanism responsible for the origin of galactic magnetic fields has probably to be looked back in the remote past, at least at a time comparable to that of galaxy formation. Furthermore, even if the galactic dynamo was effective, the origin of the seed fields which initiated the processes has still to be identified.

Even more mysterious is the origin of magnetic fields in galaxy clusters. These fields have been observed to have strength and coherence size comparable to, and in some cases larger than, galactic fields. In the standard cold dark matter (CDM) scenario of structure formation clusters form by aggregation of galaxies. It is now understood that magnetic fields in the inter-cluster medium (ICM) cannot form from ejection of the galactic fields (see Sec.1.2). Therefore, a common astrophysical origin of both types of fields seems to be excluded. Although, independent astrophysical mechanisms have been proposed for the generation of magnetic fields in galaxies and clusters, a more economical, and conceptually satisfying solution would be to look for a common cosmological origin.

Magnetic fields could have played a significant role in structure formation. It may be not a coincidence that primordial magnetic fields as those required to explain galactic fields, without having to appeal to a MHD amplification, would also produce pre-recombination density perturbations on protogalactic scales. These effects go in the right direction to solve one of the major problems of the CDM scenario of structure formation (see Sec.1.3). Furthermore, if primordial magnetic fields affected structure formation they also probably left detectable imprints in the temperature and polarization anisotropies, or the thermal spectrum, of the cosmic microwave background radiation (CMBR) (see Chap.2).

Field theory provides several clues about the physical mechanisms which may have produced magnetic fields in the early Universe. Typically, magnetogenesis requires an out-of-thermal equilibrium condition and a macroscopic parity violation. These conditions could be naturally provided by those phase transitions which presumably took place during the big-bang. Well known examples are the QCD (or quark confinement) phase transition, the electroweak (EW) phase transition, the GUT phase transition. During these transitions magnetic fields can be either generated by the turbulent motion induced in the ambient plasma by the rapid variation of some thermodynamic quantities (if the transition is first order) or by the dynamics of the Higgs and gauge fields. In the latter case the mechanism leading to magnetogenesis shares some interesting common aspects with the mechanism which have been proposed for the formation of topological defects. On the other hand, if cosmic strings were produced in the early Universe they could also generate cosmic magnetic fields in several ways. Inflation, which provides a consistent solution to many cosmological puzzles, has also several features which make it interesting in the present context (see Sec.4.5). Although to implement an inflationary scenario of magnetogenesis requires some nontrivial extensions of the particle physics standard model, recent independent developments in field theory may provide the required ingredients. Magnetic fields might also be produced by a preexisting lepton asymmetry by means of the Abelian anomaly (see Sec.4.4). Since the predictions about the strength and the spatial distribution of the magnetic fields are different for different models, the possible detection of primeval magnetic fields may shed light on fundamental physical processes which could, otherwise, be unaccessible.

Even if primordial magnetic fields did not produce any relevant effect after recombination, they may still have played a significant role in several fundamental processes which occurred in the first 100,000 years. For example, we shall show that magnetic fields may have affected the big-bang nucleosynthesis, the dynamics of some phase transitions, and baryogenesis. Since big-bang nucleosynthesis (BBN) has been often used to derive constraints on cosmological and particle physics parameters, the reader may be not surprised to learn here that BBN also provides interesting limits on the strength of primordial magnetic fields (see Chap.3). Even more interesting is the interplay which may exist between baryogenesis and magnetogenesis. Magnetic fields might have influenced baryogenesis either by affecting the dynamics of the electroweak phase transition or by changing the rate of baryon number violating sphaleron processes (see Chap.5). Another intriguing possibility is that the hypercharge component of primeval magnetic fields possessed a net helicity (Chern-Simon number) which may have been converted into baryons and leptons by the Abelian anomaly (see Chap.4). In other words, primordial magnetic fields may provide a novel scenario for the production of the observed matter-antimatter asymmetry of the Universe.

An interesting aspect of magnetic fields is their effect on the constituents of matter. This in turn is of importance on many aspects of the processes that took place in the early times. Masses of hadrons get changed so that protons are heavier than neutrons. The very nature of chirality could get changed see (Chap.5). However the characteristic field for this to happen is $H = m_\pi^2$ which is about 10^{18} G. These fields cannot exist at times when hadrons are already existing and therefore are probably not relevant. Near cosmic superconductive strings the story may be different.

Clearly, this is a quite rich and interdisciplinary subject and we will not be able to cover with the same accuracy all its different aspects. Our review is manly focused on the production mechanism and the effects of magnetic fields before, or during, the photon decoupling from matter.

In Chap.1 we shortly review the current status of the observations. In order to establish some relation between recent time and primeval magnetic fields we also provide a short description of some of the mechanisms which are supposed to control the evolution of magnetic fields in the galaxies and in the intergalactic medium. We only give a very short and incomplete description of the effect of magnetic fields on structure formation. Some basic aspects of this subject are, anyhow, presented in Chap.2 where we discuss the effect of magnetic fields on the anisotropies of the cosmic microwave background radiation. From a phenomenological point of view Chap.2 is certainly the most interesting of our review. The rapid determination of the CMBR acoustic peaks at the level of a few percent will constrain these fields significantly. We briefly touch upon the recent determination of the second acoustic peak. In Chap.3 we describe several effects of strong magnetic fields on the BBN and present some constraints which can be derived by comparing the theoretical predictions of the light elements relic abundances with observations. Since it can be of some relevance for BBN, propagation of neutrinos in magnetized media is also shortly discussed at the end of that chapter. In Chap.4 we review several models which predict the generation of magnetic fields in the early Universe. In the same chapter some possible mutual effects of magnetogenesis and baryogenesis are also discussed. Some aspects of the effects which are described in Chapts.3 and 4, which concern the stability of strong magnetic fields and the effect that they may produce on matter and gauge fields, are discussed in more details in Chap.5. At the end we report our conclusions.

Chapter 1

The recent history of cosmic magnetic fields

1.1 Observations

The main observational tracers of galactic and extra-galactic magnetic fields are (comprehensive reviews of the subject can be found in Refs.[1, 2]): the Zeeman splitting of spectral lines; the intensity and the polarization of synchrotron emission from free relativistic electrons; the Faraday rotation measurements (RMs) of polarized electromagnetic radiation passing through a ionized medium.

Typically the Zeeman splitting, though direct, is too small to be useful outside our galaxy. Unfortunately, although the synchrotron emission and RMs allows to trace magnetic fields in very distant objects, both kind of measurements requires an independent determination of the local electron density n_e. This is sometimes possible, e.g. by studying the X-ray emission from the electron gas when this is very hot, typically when this is confined in a galaxy cluster. Otherwise n_e may be not always easy to determine, especially for very rarefied media like the intergalactic medium (IGM). In the case of synchrotron emission, which intensity is proportional to $n_e B^2$, an estimation of B is sometimes derived by assuming equipartition between the magnetic and the plasma energy densities.

If the magnetic field to be measured is far away one relies on Faraday rotation. The agreement generally found between the strength of the field determined by RMs and that inferred from the analysis of the synchrotron emission in relatively close objects gives reasonable confidence on the reliability of the first method also for far away systems. It should be noted, however, that observations of synchrotron emission and RMs are sensitive to different spatial components of the magnetic field [2]. The RM of the radiation emitted by a source with redshift z_s is given by

$$RM(z_s) \equiv \frac{\Delta(\kappa)}{\Delta(\lambda^2)} = 8.1 \times 10^5 \int_0^{z_s} n_e B_{\|}(z)(1+z)^{-2} dl(z) \quad \frac{\mathrm{rad}}{\mathrm{m}^2} \tag{1.1}$$

where $B_{\|}$ is the field strength along the line of sight and

$$dl(z) = \; 10^{-6} H_0^{-1}(1+z)(1+\Omega z)^{-\frac{1}{2}} dz \;\; \mathrm{Mpc}\ . \tag{1.2}$$

H_0 is the Hubble constant. The previous expression holds for a vanishing cosmological constant and modification for finite Λ is straightforward. This method requires knowledge of the electron column and possibility of field reversals. For nearby measurements in our own galaxy pulsar frequency and their decays can pin down these effects. Otherwise these stars are too far to help. For this reason to determine the magnetic field of the IGM by Faraday RMs is quite hard and only model dependent upper limits are available.

We now briefly summarize the observational situation.

Magnetic fields in galaxies. The interstellar magnetic field in the Milky Way has been determined using several methods which allowed to obtain valuable information about the amplitude and spatial structure of the field. The average field strength is $3-4\mu$G. Such a strength corresponds to an approximate energy equipartition between the magnetic field, the cosmic rays confined in the Galaxy, and the small-scale turbulent motion [1]

$$\rho_m = \frac{B^2}{8\pi} \approx \rho_t \approx \rho_{CR}\ . \tag{1.3}$$

Remarkably, the magnetic energy density almost coincides with energy density of the cosmic microwave background radiation (CMBR). The field keeps its orientation on scales of the order of few kiloparsecs (kpc), comparable with the galactic size, and two reversals have been observed between the galactic arms, suggesting that the Galaxy field morphology may be symmetrical. Magnetic fields of similar intensity have been observed in a number of other spiral galaxies. Although equipartition fields were observed in some galaxies, e.g. M33, in some others, like the Magellanic Clouds and M82, the field seems to be stronger than the equipartition threshold. Concerning the spatial structure of the galactic fields, the observational situation is, again, quite confused with some galaxies presenting an axially symmetrical geometry, some other a symmetrical one, and others with no recognizable field structure [2].

Magnetic fields in galaxy clusters. Observations on a large number of Abel clusters [3], some of which have a measured X-ray emission, give valuable information on fields in clusters of galaxies. The magnetic field strength in the inter cluster medium (ICM) is well described by the phenomenological equation

$$B_{ICM} \sim 2\ \mu\text{G} \left(\frac{L}{10kpc}\right)^{-\frac{1}{2}} (h_{50})^{-1} \tag{1.4}$$

where L is the reversal field length and h_{50} is the reduced Hubble constant. Typical values of L are $10 - 100$ kpc which correspond to field amplitudes of $1 - 10$ μG. The concrete case of the Coma cluster [4] can be fitted with a core magnetic field $B \sim 8.3 h_{100}^{\frac{1}{2}}$ G tangled at scales of about 1 kpc. A particular example of clusters with a strong field is the Hydra A cluster for which the RMs imply a 6 μG field coherent over 100 kpc superimposed with a tangled field of strength $\sim$ 30 μG [5]. A rich set of high resolution images of radio sources embedded in galaxy clusters shows evidence of strong magnetic fields in the cluster central regions [6]. The typical central field strength $\sim 10 - 30$ μG with peak values as large as ~ 70 μG. It is noticeable that for such large fields the magnetic pressure exceeds the gas pressure derived from X-ray data suggesting that magnetic fields may play a significant role in the cluster dynamics.It is interesting, as it has been shown by Loeb and Mao [7] that a discrepancy exists between the estimate of the mass of the Abell cluster 2218 derived from gravitational lensing and that inferred from X-ray observations which can be well explained by the pressure support produced by a magnetic fields with strength ~ 50 μG. It is still not clear if the apparent decrease of the magnetic field strength in the external region of clusters is due to the intrinsic field structure or it is a spurious effect due to the decrease of the gas density. Observations show also evidence for a filamentary spatial structure of the field. According to Eilek [6] the filaments are presumably structured as a *flux rope*, that is a twisted field structure in which the field lies along the axis in the center of the tube, and becomes helical going away from the axis.

It seems quite plausible that all galaxy clusters are magnetized. As we will discuss in the next section, these observations are a serious challenge to most of the models proposed to explain the origin of galactic and cluster magnetic fields.

Magnetic fields in high redshift objects. High resolution RMs of very far quasars have allowed to probe magnetic fields in the distant past. The most significative measurements are due to Kronberg and collaborators (see Ref.[1] and Refs. therein). RMs of the radio emission of the quasar 3C191, at $z = 1.945$, presumably due a magnetized shell of gas at the same redshift, are consistent with a field strength in the range $0.4-4\ \mu$G. The field was found to maintain its prevailing direction over at least ~ 15 kpc, which is comparable with a typical galaxy size. The magnetic field of a relatively young spiral galaxy at $z = 0.395$ was determined by RMs of the radio emission of the quasar PKS 1229-021 lying behind the galaxy at $z = 1.038$. The magnetic field amplitude was firmly estimated to be in the range $1 - 4\ \mu$G. Even more interesting was the observation of field reversals with distance roughly equal to the spiral arm separation, in a way quite similar to that observed in the Milky Way.

Intergalactic magnetic fields. The radio emission of distant quasars is also used to constrain the intensity of magnetic fields in the IGM which we may suppose to pervade the entire Universe. As we discussed, to translate RMs into an estimation of the field strength is quite difficult for rarefied media in which ionized gas density and field coherence length are poorly known. Nevertheless, some interesting limits can be derived on the basis of well known estimates of the Universe ionization fraction and adopting some reasonable values of the magnetic coherence length. For example, assuming a cosmologically aligned magnetic field, as well as $\Omega = 1$, $\Lambda = 0$, and $h = 0.75$, the RMs of distant quasar imply $B_{IGM} \lesssim 10^{-11}$ G [1]. A field which is aligned on cosmological scales is, however, unlikely. As we have seen in the above, in galaxy clusters the largest reversal scale is at most 1 Mpc. Adopting this scale as the typical cosmic magnetic field coherence length and applying the RM(z_s) up to $z_s \sim 2.5$, Kronberg found the less stringent limit $B_{IGM} \lesssim 10^{-9}$ G for the magnetic field strength at present time.

A method to determine the power spectrum of cosmic magnetic fields from RMs of a large number of extragalactic sources has been proposed by Kolatt [8]. The result of this kind of analysis would be of great help to determine the origin and the time evolution of these fields.

Another interesting idea proposed by Plaga [9] is unfortunately not correct. The idea here is to look to photons from an instantaneous cosmological source, like a gamma burst or a supernova, and check for the existence of a delayed component of the signal. This new component would be due to an original photon creating an electron positron pair and in turn the charged particle sending a photon in the original direction by inverse Compton scattering. For sources at cosmological distances the delay would be sensitive to a small B field, say 10^{-11} G that would affect the motion of the charged intermediate particle. Unfortunately, the uncontrollable opening of the pair will produce a similar delay that cannot be disentangled from the time delay produced by the magnetic field.

1.2 The alternative: dynamo or primordial ?

For a long time the preferred mechanism to explain the aforementioned observations was the dynamo mechanism [10]. Today, however, new observational and theoretical results seems to point to a different scenario. Before trying to summarize the present status of art, a short, though incomplete, synthesis of what is a dynamo mechanism may be useful to some of our readers. More complete treatments of this subject can be found e.g. in Refs. [1, 11, 12, 13, 14].

A dynamo is a mechanism responsible for the conversion of kinetic energy of an electrically conducting fluid into magnetic energy. It takes place when in the time evolution equation of the magnetic field (see e.g. Ref.[15])

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{4\pi\sigma}\nabla^2\mathbf{B} \ , \tag{1.5}$$

where σ is the electric conductivity, the first term on the RHS of Eq.(1.5) (frozen-in term) dominates the second one which account for magnetic diffusion. As we will see in Sec.1.4 this statement can be reformulated in terms of the magnetic Reynolds number which has to be much larger than unity. As it is clear from Eq.(1.5), a novanishing seed field is needed to initiate the dynamo process. Other three key ingredients are generally required. They are hydrodynamic turbulence, differential rotation and fast reconnection of magnetic lines. In the frozen-in limit magnetic lines are distorted and stretched by turbulent motion. It can be shown [13] that in the same limit the ratio

B/ρ of the magnetic field strength with the fluid density behaves like the distance between two fluid elements. As a consequence, a stretching of the field lines result in an increase of B. However, this effect alone would not be sufficient to explain the exponential amplification of the field generally predicted by the dynamo advocates. In fact, turbulence and global rotation of the fluid (e.g. by Coriolis force) may produce twisting of closed flux tubes and put both part of the twisted loop together, restoring the initial single-loop configuration but with a double flux (see Fig.2 in Ref.[12]). The process can be iterated leading to a 2^n-amplification of the magnetic field after the n-th cycle. The merging of magnetic loops, which produce a change in the topology (quantified by the so called magnetic helicity, see Sec.1.4) of the magnetic field lines, requires a finite, though small, resistivity of the medium. This process occurs in regions of small extension where the field is more tangled and the diffusion time is smaller (see Sec.1.4). As a consequence, the entire magnetic configuration evolves from a small-scale tangled structure towards a mean ordered one.

The most common approach to magnetic dynamo is the so called mean field dynamo. It is based on the assumption that fluctuations in the magnetic and velocity fields are much smaller of the mean slowly varying components of the corresponding quantities. Clearly, mean field dynamo is suitable to explore the amplification of large scale magnetic structures starting from small scale seed fields in the presence of a turbulent fluid. The temporal evolution of the mean component of the magnetic field is obtained by a suitable averaging of Eq.(1.5) (below, mean quantities are labelled by a 0 and random quantities by a 1)

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\alpha \mathbf{B}_0 + \mathbf{v}_0 \times \mathbf{B}_0) - \nabla \times [(\eta + \beta) \nabla \times \mathbf{B}_0] \ , \tag{1.6}$$

where

$$\alpha = -\frac{1}{3}\tau_c \langle \mathbf{v}_1 \cdot \nabla \times \mathbf{v}_1 \rangle \qquad \beta = \frac{1}{3}\tau_c \langle \mathbf{v}_1^2 \rangle \ , \tag{1.7}$$

$\eta = 1/4\pi\sigma$ is the magnetic diffusivity, and τ_c is the correlation time for the ensemble of random velocities. The coefficient α is proportional to the helicity $h = \langle \mathbf{v}_1 \cdot \nabla \times \mathbf{v}_1 \rangle$ of the flow; h measures the degree to which streamlines are twisted. A macroscopic parity violation is required to have $\alpha \propto h \neq 0$. One of the possible sources of this violation can be the Coriolis force produced by the rotation of the galaxy [11]. The term $\nabla \times (\beta \nabla \times \mathbf{B}_0)$ describes the additional field dissipation due to turbulent motion. Turbulence plays another crucial role in the generation of a toroidal component of the large scale magnetic fields which is essential for the stability of the entire field configuration [13]. Indeed the helicity, through the α-term, is responsible for the generation of an electric field parallel to $\mathbf{B}_0$ [1]. This field provides a mode for conversion of toroidal into poloidal magnetic field components. This is the so called α-effect. To complete the "dynamo cycle" $B_T \rightleftharpoons B_P$, another mechanism is required to convert the poloidal component into a toroidal one. This mechanism is provided by the differential rotation of the galactic disk which will wrap up the field line producing a toroidal field starting form a poloidal component; this is the ω-effect. The combination of the α and ω effects gives rise to the, so called, $\alpha - \omega$ galactic dynamo. As a result of the coexistence of the poloidal and toroidal magnetic components, one of the main prediction of the of $\alpha - \omega$ dynamo is the generation of an axially symmetric mean fields.

In the case the β term can be neglected, the solution of the mean field dynamo equation (1.6) can be written in the form [10]

$$\mathbf{B}_0 = (\pm \sin kz, \cos kz, 0)\, e^{\gamma t} \ , \tag{1.8}$$

where z is the coordinate along the galaxy rotation axis, and $\gamma = -\eta k^2 \pm \alpha k$, $k \sim 1/L$ being the wavenumber. The field grows exponentially with time for non-zero helicity and if the scale L is sufficiently large. A general prediction of a dynamo mechanism is that amplification ends when equipartition is reached between the kinetic energy density of the small-scale turbulent fluid motion and the magnetic energy density. This correspond to a magnetic field strength in the range of $2 - 8\ \ \mu$G. Depending on the details of the model and of the local properties of the medium, the time required to reach saturation, starting from a seed magnetic fields with intensity as low as 10^{-20} G, may be of $10^8 - 10^9$ years. It should be noted that such an estimation holds under the assumption that the Universe is dominated by CDM with no cosmological constant. If, however, as recent observations of distant type-IA supernovae [16] and CMB anisotropy measurments [17] suggest, the Universe posses a sizeable cosmological constant, the available time for the dynamo amplification increases and a smaller initial seed field may be required. This point has been recently rised by Davis, Lilley and Törnkvist [18] who showed as the required seed field might be as low as 10^{-30} G.

[1] Our readers with some experience in field theory may recognize that by producing parallel electric and magnetic fields the α term is responsible of a sort of macroscopic CP violation.

In the last decade the effectiveness of the mean field dynamo has been questioned by several experts of the field (for a recent review see Ref.[14]). One of the main arguments rised by these authors against this kind of dynamo is that it neglects the strong amplification of small-scale magnetic fields which reach equipartition, stopping the process, before a coherent field may develop on galactic scales.

The main, though not the unique, alternative to the galactic dynamo is to assume that the galactic field results directly from a primordial field which gets adiabatically compressed when the protogalactic cloud collapse. Indeed, due to the large conductivity of the intergalactic medium (see Sec.1.4), magnetic flux is conserved in the intergalactic medium which implies that the magnetic field has to increase like the square of the size of the system l. It follows that

$$B_{\rm prim,\ 0} = B_{\rm gal} \left(\frac{\rho_{\rm cosmic}}{\rho_{\rm gal}}\right)^{2/3} . \tag{1.9}$$

Since the present time ratio between the interstellar medium density in the galaxies and the density of the IGM is $\rho_{\rm IGM}/\rho_{\rm gal} \simeq 10^{-6}$, and $B_{\rm gal} \sim 10^{-6}$ G, we see that the required strength of the cosmic magnetic field at the galaxy formation time ($z \sim 5$), adiabatically rescaled to present time, is

$$B_{\rm prim,\ 0} \simeq 10^{-10}\ \ {\rm G}. \tag{1.10}$$

This value is compatible with the observational limit on the field in the IGM derived by RMs, with the big-bang nucleosynthesis constraints (see Chap.3), and may produce observable effects on the anisotropies of the cosmic microwave background radiation (see Chap.2). Concerning the spatial structure of the galactic field produced by this mechanism, differential rotation should wrap the field into a symmetric spiral with field reversal along the galactic disk diameter and no reversal across the galactic plane [2].

To decide between the dynamo and the primordial options astrophysicists have to their disposal three kind of information. They are:

- the observations of intensity and spatial distribution of the galactic magnetic fields;
- the observations of intensity and spatial distribution of the intergalactic magnetic fields;
- the observations of magnetic fields in objects at high redshift.

Observations of the magnetic field intensity in a some galaxies, including the Milky Way, show evidence of approximate equipartition between turbulent motion and magnetic energies, which is in agreement with the prediction of linear dynamo. There are however some exceptions, like the M82 galaxy and the Magellanic Clouds, where the field strength exceed the equipartition field. An important test concerns the parity properties of the field with respect to the rotations by π about the galactic center. As we have discussed above, the primordial theory predicts odd parity and the presence of reversals with radius (a symmetric spiral field), whereas most dynamo models predict even parity (axially symmetric spiral) with no reversal. Although most galaxies exhibit no recognizable large-scale pattern, reversals are observed between the arms in the Milky Way, M81 and the high redshift galaxy discussed in the previous section, though not in M31 and IC342. Given the low statistical significance of the sample any conclusions are, at the moment, quite premature [2].

As we reported in the previous section only upper limits are available for the intensity of magnetic fields in the intergalactic medium. Much richer is the information that astrophysicists collected in the recent years about the magnetic fields in the inter-cluster medium (ICM). As we have seen, magnetic fields of the order of $1-10$ μG seems to be a common features of galaxy clusters. The strength of these fields is comparable to that of galactic fields. This occurs in spite of the lower matter density of the ICM with respect to the density of interstellar medium in the galaxies. It seems quite difficult to explain the origin of the inter-cluster magnetic fields by simple ejection of the galactic fields. Some kind of dynamo process produced by the turbulent wakes behind galaxies moving in the ICM has been proposed by some authors but criticized by some others (for a review see Ref.[1]). This problem has become even more critical in light of recent high-precision Faraday RMs which showed evidence of magnetic fields with strength exceeding 10 μG in the cluster central regions. According to Kronberg [1], the observed independence of the field strength from the local matter density seems to suggest that galactic systems have evolved in a magnetic environment where $B \gtrsim 1$ μG. This hypothesis seems to be corroborated by the measurements of the Faraday rotations produced by high redshift protogalactic clouds. As we wrote in the previous section such measurements show evidence for magnetic fields of the order of 1 μG in clouds with redshift larger than 1. Since, at that time galaxies should have rotated few times, these observations pose a challenge to the galactic dynamo advocates. We should keep in mind, however, that galaxy formation in the presence of magnetic fields with strength $\gtrsim 10^{-8}$ G may be problematic due to the magnetic pressure which inhibits the collapse [19].

It is worthwhile to observe that primordial (or pre-galactic) magnetic fields are not necessarily produced in the early Universe, i.e. before recombination time. Several alternative astrophysical mechanisms have been proposed like the generation of the fields by a Biermann battery effect [20] (see also Ref.[1]). It has been suggested that the Biermann battery may produce seed fields which are successively amplified on galactic scale by a dynamo powered by the turbulence in the protogalactic cloud [14, 21]. This mechanism, however, can hardly account for the magnetic fields observed in the galaxy clusters. Therefore, such a scenario would lead us to face an unnatural situation where two different mechanisms are invoked for the generation of magnetic fields in galaxies and clusters, which have quite similar characteristics and presumably merge continuously at the border of the galactic halos.

Another possibility is that magnetic fields may have been generated by batteries powered by starbursts or jet-lobe radio sources (AGNs). In a scenario recently proposed by Colgate and Li [22] strong cluster magnetic fields are produced by a dynamo operating in the accretion disk of massive black holes powering AGNs. We note, however, that the dynamics of the process leading to the formation of massive black holes is still unclear and that preexisting magnetic fields may be required to carry away the huge angular moment of the in-falling matter (see e.g. Ref.[19]). For the same reason, preexisting magnetic fields may also be required to trigger starbursts (see the end of next section). This suggests that seed fields produced before recombination time may anyway be required.

In conclusion, although the data available today do not allow to answer yet to the question raised in this section, it seems that recent observations and improved theoretical work are putting in question the old wisdom in favour of a dynamo origin of galactic magnetic fields. Especially the recent observations of strong magnetic fields in galaxy clusters suggest that the origin of these fields may indeed be primordial.

Furthermore, magnetic fields with strength as large as that required for the primordial origin of the galactic fields through gravitational compression of the magnetized fluid, should give rise to interesting, and perhaps necessary, effects for structure formation. This will be the subject of the next section.

1.3 Magnetic fields and structure formation

The idea that cosmic magnetic fields may have played some role in the formation of galaxies is not new. Some early work has been done on this subject, e.g. by Peblees [23], Rees and Rheinhardt [24] and especially by Wasserman [25]. A considerable amount of recent papers testify the growing interest around this issue. A detailed review of this particular aspect of cosmology is, however, beyond the purposes of this report. We only summarize here few main points with the hope to convince the reader of relevance of this subject.

Large scale magnetic fields modify standard equations of linear density perturbations in a gas of charged particles by adding the effect of the Lorentz force. In the presence of the field the set of Euler, continuity and Poisson equations become respectively [25]

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\frac{\dot{a}}{a}\mathbf{v}+\frac{\mathbf{v}\cdot\nabla\mathbf{v}}{a}\right) = -\frac{\nabla \mathbf{p}}{a}-\rho\frac{\nabla\phi}{a}+\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi a}\ , \tag{1.11}$$

$$\frac{\partial\delta}{\partial t}+3\frac{\dot{a}}{a}\rho+\frac{\nabla\cdot(\rho\mathbf{v})}{a} = 0\ , \tag{1.12}$$

$$\nabla^2\phi = 4\pi G a^2\left(\rho-\rho_0(t)\right)\ . \tag{1.13}$$

Here a is the scale factor and other symbols are obvious. This set of equations is completed by the Faraday equation

$$\frac{\partial(a^2\mathbf{B})}{\partial t}=\frac{\nabla\times(\mathbf{v}\times a^2\mathbf{B})}{a}\ , \tag{1.14}$$

and

$$\nabla\cdot\mathbf{B}=0\ . \tag{1.15}$$

The term due to the Lorentz force is clearly visible in the right hand side of the Euler equation. It is clear that, due to this term, an inhomogeneous magnetic field becomes itself a source of density, velocity and gravitational perturbations in the electrically conducting fluid. It has been estimated [25] that the magnetic field needed to produce a density contrast $\delta\sim 1$, as required to induce structure formation on a scale l, is

$$B_0(l)\sim 10^{-9}\left(\frac{l}{1\ \mathrm{Mpc}}\right)h^2\Omega\quad \mathrm{G}\ . \tag{1.16}$$

In his recent book Peebles Ref.[26] pointed-out a significabt coincidence: the primordial magnetic field required to explain galactic fields without invoking dynamo amplification (see Eq.1.10) would also play a relevant dynamical role in the galaxy formation process.

The reader may wonder if such a dynamical role of magnetic fields is really required. To assume that magnetic fields were the dominant dynamical factor at the time of galaxy formation and that they were the main source of initial density perturbations is perhaps too extreme and probably incompatible with recent measurements of the CMBR anisotropies. A more reasonable possibility is that magnetic fields are an important missing ingredient in the current theories on large scale structure formation (for a recent review on this subject see Ref.[27]). It has been argued by Coles [28] that an inhomogeneous magnetic field could modulate galaxy formation in the cold dark matter picture (CDM) by giving the baryons a streaming velocity relative to the dark matter. In this way, in some places the baryons may be prevented from falling into the potential wells and the formation of luminous galaxies on small scales may be inhibited. Such an effect could help to reconcile the well know discrepancy of the CDM model with clustering observations without invoking more exotic scenarios.

Such a scenario received some support from a paper by Kim, Olinto and Rosner [29] which extended Wasserman's [25] pioneering work. Kim et al. determined the power spectrum of density perturbation due to a primordial inhomogeneous magnetic field. They showed that a present time *rms* magnetic field of 10^{-10} G may have produced perturbations on galactic scale which should have entered the non-linear grow stage at $z \sim 6$, which is compatible with observations. Although, Kim et al. showed that magnetic fields alone cannot be responsible of the observed galaxy power spectrum on large scales, according to the authors it seems quite plausible that in a CDM scenario magnetic fields played a not minor role by introducing a bias for the formation of galaxy sized objects.

A systematic study of the effects of magnetic fields on structure formation was recently undertaken by Battaner, Florido and Jimenez-Vicente [30], Florido and Battaner [31], and Battaner, Florido and Garcia-Ruiz [32]. Their results show that primordial magnetic fields with strength $B_0 \lesssim 10^{-9}$ in the pre-recombination era are able to produce significant anisotropic density inhomogeneities in the baryon-photon plasma and in the metric. In particular, Battaner at al. showed that magnetic fields tend to organize themselves and the ambient plasma into filamentary structures. This prediction seems to be confirmed by recent observations of magnetic fields in galaxy clusters [6]. Battaner et al. suggest that such a behavior may be common to the entire Universe and be responsible for the very regular spider-like structure observed in the local supercluster [33] as for the filaments frequently observed in the large scale structure of the Universe [27]. Araujo and Opher [34] have considered the formation of voids by the magnetic pressure.

An interesting hypothesis has been recently rised by Totani [35]. He suggested that spheroidal galaxy formation occurs has a consequence of starbursts triggered by strong magnetic fields. Totani argument is based on two main observational facts. The first is that magnetic fields strengths observed in spiral galaxies sharply concentrate at few microgauss (see Sec.1.1), quite independently on the galaxy luminosity and morphology. The second point on which Totani based his argument, is that star formation activity has been observed to be correlated to the strength of local magnetic field [36]. A clear example is given by the spiral galaxy M82, which has an abnormally large magnetic field of $\sim 10\ \mu$G and is known as an archetypal starburst galaxy. Such a correlation is theoretical motivated by the so-called *magnetic braking* [19]: in order for a protostellar gas cloud to collapse into a star a significant amount of angular moment must be transported outwards. Magnetic fields provide a way to fulfill this requirement by allowing the presence of Alfvén waves (see Sec.2.2) which carry away the excess of angular moment. Whereas it is generally agreed that galaxy bulges and elliptical galaxies have formed by intense starburst activity at high redshift, the trigger mechanism leading to this phenomenon is poorly known. According to Totani, starbursts, hence massive galaxy formation, take place only where the magnetic field is stronger of a given threshold, which would explain the apparent uniformity in the magnetic field amplitude in most of the observed galaxies. The value of the threshold field depends on the generation mechanism of the galactic magnetic field. Totani assumed that a seed field may have been produced by a battery mechanism followed by a dynamo amplification period. Such an assumption, however, looks not necessary and a primordial field may very well have produced the same final effect.

1.4 The evolution of primordial magnetic fields

A crucial issue for the investigation of a possible primordial origin of present time galactic and intergalactic magnetic fields is that concerning the time evolution of the magnetic fields in the cosmic medium. Three conditions are needed for the persistence of large static fields:

a) intrinsic stability of the field;

b) the absence of free charges which could screen the field;

c) to have a small diffusion time of the field with respect to the age of the Universe.

Condition a) does not hold for strong electric fields. It is a firm prediction of QED that an electric field decays by converting its energy in electron-positron pairs if $e|E| \geq m_e^2$ [37, 38]. This, however, is a purely electric phenomenon. Although, at the end of the sixties, there was a claim that strong magnetic fields may decay through a similar phenomenon [39] the argument was proved to be incorrect. Only very strong fields may produce nontrivial instabilities in the QCD (if $B > 10^{17}$ G) and the electroweak vacuum (if $B > 10^{23}$ G) which may give rise to a partial screening of the field. These effects (see Chap.5) may have some relevance for processes which occurred at very early times and, perhaps, for the physics of very peculiar collapsed objects like magnetars [40]. They are, however, irrelevant for the evolution of cosmic magnetic fields after BBN time. The same conclusion holds for finite temperature and densities effects which may induce screening of static magnetic fields (see e.g. Ref.[41]).

Condition b) is probably trivially fulfilled for magnetic fields due to the apparent absence of magnetic monopoles in nature. It is interesting to observe that even a small abundance of magnetic monopoles at present time would have dramatic consequences for the survival of galactic and intergalactic magnetic fields which would lose energy by accelerating the monopoles. This argument was first used by Parker [42] to put a severe constraint on the present time monopole flux, which is $F_M \lesssim 10^{-15}$ cm$^{-2}s^{-1}$sr^{-1}. It was recently proposed by Kephart and Weiler [43] that magnetic monopoles accelerated by galactic magnetic fields could give rise to the highest energy cosmic rays ($E \lesssim 10^{19}$ eV) and explain the violation of the famous Greisen-Zatsepin-Kuzmin cut-off [44].

Also the condition c) does not represent a too serious problem for the survival of primordial magnetic fields. The time evolution law of a magnetic field in a conducting medium has been already written in Eq.(1.5).

Neglecting fluid velocity this equation reduces to the diffusion equation which implies that an initial magnetic configuration will decay away in a time

$$\tau_{\rm diff}(L) = 4\pi\sigma L^2 \ , \tag{1.17}$$

where L is the characteristic length scale of the spatial variation of $\mathbf{B}$. In a cosmological framework, this means that a magnetic configuration with coherence length L_0 will survive until the present time t_0 ($t = 0$ corresponds to the big-bang time) only if $\tau_{\rm diff}(L_0) > t_0$. In our convention, L_0 corresponds to the present time length scale determined by the Hubble law

$$L_0 = L(t_i)\, \frac{a(t_0)}{a(t_i)} \ , \tag{1.18}$$

where $a(t)$ is the Universe scale factor and $L(t_i)$ is the length scale at the time at which the magnetic configuration was formed. Note that L_0 may not coincide with the actual size of the magnetic configuration since other effects (see below) may come-in to change the comoving coherence length. As we see from Eq.(1.17) the relevant quantity controlling the decay time of a magnetic configuration is the electric conductivity of the medium. This quantity changes in time depending on the varying population of the available charge carriers and on their kinetics energies. However, since most of the Universe evolution takes place in a matter dominated regime, during which all charge carriers are non-relativistic, our estimate of the magnetic diffusion length is simpler. In general, electric conductivity can be determined by comparing Ohm law $\mathbf{J} = \sigma\mathbf{E}$ with the electric current density definition $\mathbf{J} = ne\mathbf{v}$, where for simplicity we considered a single charge carrier type with charge e, number density n and velocity $\mathbf{v}$. The mean drift velocity in the presence of the electric field $\mathbf{E}$ is $\mathbf{v} \sim e\mathbf{E}\tau/m$ where m is the charge carrier mass and τ is the average time between particle collisions. Therefore the general expression for the electron conductivity is [2]

$$\sigma = \frac{ne^2\tau}{m} \ . \tag{1.19}$$

After recombination of electron and ions into stable atoms Universe conductivity is dominated by residual free electrons. Their relative abundance is

[2] In the case the average collision time of the charge carrier is larger than the Universe age τ_U, the latter has to be used in place of τ in Eq.1.19 [45].

roughly determined by the value that this quantity took at the time when the rate of the reaction $p + e \leftrightarrow H + \gamma$ became smaller than the Universe expansion rate. In agreement with the results reported in Ref.[46], we use

$$n_e(z) \simeq 3 \times 10^{-10}\ \mathrm{cm}^{-3}\ \Omega_0 h\ (1+z)^3\ , \tag{1.20}$$

where Ω_0 is the present time density parameter and h is Hubble parameter. Electron resistivity is dominated by Thomson scattering off cosmic background photons. Therefore $\tau \simeq 1/n_\gamma \sigma_T$, where $\sigma_T = \dfrac{e^4}{6\pi m_e^2}$ is the Thomson cross section, and $n_\gamma = 4.2 \times 10^2 (1+z)^3$. Substituting these expressions in Eq.(1.19) we get

$$\sigma = \frac{n e^2}{m_e n_\gamma \sigma_T} \simeq 10^{11} \Omega_0 h\ \ s^{-1}. \tag{1.21}$$

It is noticeable that after recombination time Universe conductivity is a constant. Finally, the cosmic diffusion length, i.e. the minimal size of a magnetic configuration which can survive diffusion during the Universe time-life t_0, is found by substituting $t_0 = 2 \times (\Omega_0 h^2)^{-1/2}\ \ \mathrm{s}^{-1}$ into Eq.(1.17) which, adopting $\Omega_0 = 1$ and $h = 0.6$, gives

$$L_{\rm diff} \simeq 2 \times 10^{13}\ \mathrm{cm} \simeq 1\ \mathrm{A.U.} \tag{1.22}$$

It follows from this result that magnetic diffusion is negligible on galactic and cosmological scales.

The high conductivity of the cosmic medium has other relevant consequences for the evolution of magnetic fields. Indeed, as we already mentioned in the Introduction, the magnetic flux through any loop moving with fluid is a conserved quantity in the limit $\sigma \to \infty$. More precisely, it follows from the diffusion equation (1.5) and few vector algebra operations (see Ref.[15])that

$$\frac{d\Phi_S(B)}{dt} = -\frac{1}{4\pi\sigma} \int_S \nabla \times (\nabla \times \mathbf{B}) \cdot d\mathbf{S}\ , \tag{1.23}$$

where S is any surface delimited by the loop. On scale where diffusion can be neglected the field is said to be *frozen-in*, in the sense that lines of force move together with the fluid. Assuming that the Universe expands isotropically [3],

[3] In the Sec.2.1 we shall discuss under which hypothesis such an assumption is consistent with the presence of a cosmic magnetic field.

and no other effects come-in, magnetic flux conservation implies

$$B(t) = B(t_i) \left(\frac{a(t_i)}{a(t)}\right)^2 \quad . \tag{1.24}$$

This will be one of the most relevant equations in our review. It should be noted by the reader that $B(t)$ represents the local strength of the magnetic field obtained by disregarding any effect that may be produced by spatial variations in its intensity and direction. Eq.(1.24) is only slightly modified in the case the a uniform magnetic field produce a significative anisotropic component in the Universe expansion (see Sec.2.1).

Another quantity which is almost conserved due to the high conductivity of the cosmic medium is the, so-called, *magnetic helicity*, defined by

$$\mathcal{H} \equiv \int_V d^3x \, \mathbf{B} \cdot (\nabla \times \mathbf{B}) \quad . \tag{1.25}$$

Helicity is closely analogous to vorticity in fluid dynamics. In a field theory language, $\mathcal{H}$ can be identified with the Chern-Simon number which is known to be related to the topological properties of the field. Indeed, it is known from Magnetohydrodynamics (MHD) that $\mathcal{H}$ is proportional to the sum of the number of links and twists of the magnetic field lines [47]. As it follows from Eq.(1.5), the time evolution of the magnetic helicity is determined by

$$\frac{d\mathcal{H}}{dt} = -\frac{1}{4\pi\sigma} \int_V d^3x \mathbf{B} \cdot (\nabla \times \mathbf{B}) \quad . \tag{1.26}$$

As we shall show in Chap.4, several models proposed to explain the origin of primordial magnetic fields predict these fields to have some relevant amount of helicity.

In the previous section we have already mentioned the important role played by magnetic helicity in some MHD dynamo mechanisms driving an instability of small scale magnetic fields into large scale fields. A similar effect may take place at a cosmological level leading to significative corrections to the simple scaling laws expressed by the Eqs.(1.18,1.24). Generally, these kind of MHD effects occur in the presence of some, turbulent motion of the conductive medium (note that Eqs.(1.18,1.24) has been derived under the assumption of vanishing velocity of the fluid $v = 0$). Hydrodynamic turbulence is generally parameterized in terms of the Reynolds number, defined by

$$Re = \frac{vL}{\nu} \ , \tag{1.27}$$

where ν is the kinematic viscosity. Fluid motion is said to be turbulent if $Re \gg 1$. In the presence of a magnetic field another parameter turns-out to be quite useful. This is the *magnetic Reynolds number* defined by

$$Re_M = \frac{vL}{\eta} \ , \tag{1.28}$$

where $\eta = \frac{1}{4\pi\sigma}$. When $Re_M \gg 1$ transport of the magnetic lines with the fluid dominates over diffusion. In this case hydrodynamic turbulence in a conducting medium give rise to *magnetic turbulence*. It is often assumed in MHD that a fully developed magnetic turbulence give rise to equipartion between the kinetic and the magnetic energy of the fluid. Whether equipartition hypothesis is valid or not is a controversial issue.

Both the hydrodynamic and magnetic Reynolds numbers can be very large in the early Universe. This is a consequence of the high electric conductivity and low viscosity of the medium and, especially, of the large scales which are involved. The electric conductivity of the early Universe has been computed by several authors. A first simple estimation of σ in the radiation dominated era was performed by Turner and Widrow [45]. In terms of the resistivity η their result is $\eta \sim \frac{\alpha}{T}$. A more exact series of calculations can be found in Ref.[48] which include logarithmic corrections due to Debye and dynamical screening. As a result a more correct expression for η is

$$\eta \sim \frac{\alpha}{T} \ln(1/\alpha) \ . \tag{1.29}$$

Other detailed computations of the Universe conductivity close to the QCD and the electroweak phase transitions are available in Refs.[49]. The kinematic viscosity follows the behavior [50]

$$\nu \sim \frac{1}{\alpha T \ln(1/\alpha)} \ . \tag{1.30}$$

In the early Universe $\nu \gg \eta$, i.e. $Re_M \gg Re$. Concerning the absolute value of these parameters, using the previous expressions it is easy to verify that for a reasonable choice of the velocity field that may be produced by a phase transition, $v \lesssim 10^{-3}$, both Re and Re_M are much larger than unity by several orders of magnitude, even for very small scales (for more details see Chap.4). It seems that the early Universe was a quite "turbulent child" ! Turbulence is expected to cease after e^+e^- annihilation since this process reduces the plasma electron population and therefore increases the photon diffusion length hence also the kinematic viscosity. This should happen at a temperature around 1 MeV.

Turbulence is expected to produce substantial modification in the scaling laws of cosmological magnetic fields. This issue has been considered by several authors. Brandenburg, Enqvist and Olesen [51] first consider MHD in an expanding Universe in the presence of hydro-magnetic turbulence. MHD equations were written in a covariant form and solved numerically under some simplifying assumptions. The magnetic field was assumed to be distributed randomly either in two or three spatial dimensions. In the latter case a cascade (shell) model was used to reduce the number of degree of freedom. In both cases a transfer of magnetic energy from small to large magnetic configurations was observed in the simulations. In hydrodynamics this phenomenon is know as an *inverse cascade*. Cascade processes are know to be related to certain conservation properties that the basic equations obey [52]. In the two-dimensional inverse cascade, the relevant conserved quantity is the volume integral of the vector potential squared, $\int d^2x \mathbf{A}^2$, whereas in the three-dimensional cases it is the magnetic helicity. It was recently showed by Son [50] that no inverse cascade can develop in 3d if the mean value of $\mathcal{H}$ vanishes. If this is the case, i.e. in the presence of non-helical MHD turbulence, there is still an anomalous grow of the magnetic correlation length with respect to the scaling given in Eq.(1.18) but this is just an effect of a *selective decay* mechanism: modes with larger wavenumbers decay faster than those whose wavenumbers are smaller. Assuming Universe expansion is negligible with respect to the decay time, which is given by the eddy turnover time $\tau_L \sim L/v_L$, and the decay of the large wavenumber modes does not affect those with smaller wavenumbers, Son found that the correlation length scale with time as

$$L(t) \sim \left(\frac{t}{t_i}\right)^{2/5} L_i \tag{1.31}$$

where $\tau_i = L_i/v_i$ is the eddy turnover time at $t = 0$. Assuming equipartition of the kinetic and magnetic energies, that is $v_L \sim B_L$, it follows that the energy decay with time like $t^{-6/5}$. When the Universe expansion becomes not negligible, i.e. when $t > t_0$, one has to take into account that the correlation length grows as $\tau^{2/5}$, where τ is the conformal time. Since $\tau \sim T^{-1}$, it follows the $T^{-2/5}$ law. In the real situation, the final correlation length at the present epoch is,

$$L_0 = L_i \cdot \left(\frac{t_0 v_i}{L_i}\right)^{2/5} \cdot \left(\frac{T_i}{T_{\rm nt}}\right)^{2/5} \cdot \frac{T_i}{T_0} \, . \tag{1.32}$$

In the above, the first factor comes from the growth of the correlation length in the time interval $0 < t < t_0$ when eddy decay is faster than Universe expansion; the second factor comes from the growth of L in the $t > t_0$ period; the last factor comes from trivial redshift due to the expansion of the Universe. $T_{\rm nt}$ is the temperature of the Universe when the fluid becomes non-turbulent. As we discussed $T_{\rm nt} \sim 1$ MeV. If, for example, we assume that turbulence was produced at the electroweak phase transition, so that $T_i = T_{EW} \sim 100$ GeV, that $v_i \sim 0.1$ and $L_i \sim 10^{-2} r_H(T_{EW}) \sim 10^{-2}$ cm, one find $L_0 \sim 100$ AU. This result has to be compared with the scale one would have if the only mechanism of dissipation of magnetic energy is resistive diffusion which, as we got in Eq.(1.22) is ~ 1 AU.

A larger coherence length can be obtained by accounting for the magnetic helicity which is probably produced during a primordial phase transition. The conservation of $\mathcal{H}$ has an important consequence for the evolution of the magnetic field. When $\mathcal{H}$ is non-vanishing, the short-scale modes are not simply washed out during the decay: their magnetic helicity must be transferred to the long-scale ones. Along with the magnetic helicity, some magnetic energy is also saved from turbulent decay. In other words, an inverse cascade is taking place. Assuming maximal helicity, i.e. that $\mathbf{B} \cdot (\nabla \times \mathbf{B}) \sim LB^2$, the conservation of this quantity during the decay of turbulence implies the scaling law

$$B_L \sim B_i \left(\frac{L}{L_i}\right)^{-1/2} \, .$$

This corresponds to "line averaging", which gives a much larger amplitude of the magnetic field than the usual "volume averaging". Equipartition between magnetic and kinetic energy implies

$$v_L \sim v_i \left(\frac{L}{L_i}\right)^{-1/2} \, .$$

This relation together with the expression for the eddy decay time, $\tau_L = L/v_L$, leads to the following scaling law for the correlation length of helical magnetic structures

$$L \sim L_i \left(\frac{t}{t_i}\right)^{2/3} \quad . \tag{1.33}$$

Comparing this result with Eq.(1.31), we see that in the helical case the correlation length grow faster than it does in the turbulent non-helical case. The complete expression for the scaling of L is finally obtained by including trivial redshift into Eq.(1.33). Since in the radiation dominated era $T^{-1} \sim a \sim t^{1/2}$, we have [50]

$$L_0 = \left(\frac{T_0}{T_{\rm nt}}\right) \left(\frac{T_i}{T_0}\right)^{5/3} L_i \ , \tag{1.34}$$

and

$$B_0 = \left(\frac{T_{\rm nt}}{T_0}\right)^{-2} \left(\frac{T_i}{T_0}\right)^{-7/3} B(T_i) \ . \tag{1.35}$$

According to Son, [50] helical hydromagnetic turbulence survives longer than non-helical turbulence allowing $T_{\rm nt}$ to be as low as 100 eV. If again we assume that helical magnetic turbulence is generated at the electroweak phase transition (which will be justified in Chap.4) we find

$$L_0 \sim L_i \left(\frac{T_{EW}}{T_{\rm nt}}\right)^{5/3} \left(\frac{T_{\rm nt}}{T_0}\right) \sim 100 \ \ {\rm pc} \ , \tag{1.36}$$

which is much larger than the result obtained in the non-helical case. It is worthwhile to observe that,as the scale derived in the previous expression is also considerably larger than the cosmological magnetic diffusion length scale given in Eq.(1.22), that magnetic field fields produced by the EW phase transition may indeed survive until present.

Chapter 2

Effects on the Cosmic Microwave Background

2.1 The effect of a homogeneous magnetic field

It is well know from General Relativity that electromagnetic fields can affect the geometry of the Universe. The energy momentum tensor

$$T^{\alpha\beta}_{\rm em} = \frac{1}{4\pi}\left(-F^{\alpha\mu}F^{\beta}_{\mu} + \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right) , \tag{2.1}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor, acts as a source term in the Einstein equations. In the case of a homogeneous magnetic field directed along the z-axis

$$T^{00} = T^{11} = T^{22} = -T^{33} = \rho_B = \frac{B^2}{8\pi} \qquad T^{0i} = 0 \; . \tag{2.2}$$

Clearly, the energy-momentum tensor becomes anisotropic due to the presence of the magnetic field. There is a positive pressure term along the x and y-axes but a "negative pressure" along the field direction. It is known that an isotropic positive pressure increases the deceleration of the universe expansion while a negative pressure tends to produce an acceleration. As a consequence, an anisotropic pressure must give rise to an anisotropy expansion law [53].

Cosmological models with a homogeneous magnetic field have been considered by several authors (see e.g. [54]). To discuss such models it is beyond the purposes of this review. Rather, we are more interested here in the possible signature that the peculiar properties of the space-time in the presence of a cosmic magnetic field may leave on the Cosmic Microwave Background Radiation (CMBR).

Following Zeldovich and Novikov [53] we shall consider the most general axially symmetric model with the metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2) - b^2(t)dz^2 \ . \tag{2.3}$$

It is convenient to define $\alpha = \frac{\dot{a}}{a}$; $\beta = \frac{\dot{b}}{b}$; and

$$r \equiv \frac{\rho_B}{\rho_{\rm rad}} \qquad \sigma \equiv \alpha - \beta \ . \tag{2.4}$$

Then, assuming $r, \sigma < 1$, the Einstein equations are well approximated by

$$\frac{d}{dt}\left(\frac{\sigma}{H}\right) = -\left(\frac{\sigma}{H}\right)\frac{\gamma-2}{\gamma t} + \frac{4r}{\gamma t} \tag{2.5}$$

$$\frac{dr}{dt} = -\frac{2r}{9\gamma t}\left(4\frac{\sigma}{H} + 9\gamma - 12\right) \ , \tag{2.6}$$

where $H = (2\alpha+\beta)$ and γ are defined by the equation of state $p = (\gamma-1)\rho$. It is easy to infer from the first of the previous equations that the magnetic field acts so as to conserve the anisotropy that would otherwise decay with time in the case $r = 0$. By substituting the asymptotic value of the anisotropy, i.e. $\sigma \to 6r$, into the evolution equation for r in the RD era one finds

$$r(t) = \frac{q}{1 + 4q\ln(t/t_0)} \ , \tag{2.7}$$

where q is a constant. Therefore, in the case the cosmic magnetic field is homogeneous, the ratio of the magnetic and blackbody radiation densities is not a constant, but falls logarithmically during the radiation era.

In order to determine the temperature anisotropy of the CMBR we assume that at the recombination time $t_{\rm rec}$ the temperature is everywhere $T_{\rm rec}$. Then, at the present time, t_0, the temperature of relic photons coming from the x (or y) and z directions will be respectively

$$T_{x,y} = T_{\rm rec}\frac{a}{a_0} = T_{\rm rec}\exp\left(-\int_{t_{\rm rec}}^{t_0}\alpha dt\right) \qquad T_z = T_{\rm rec}\frac{b}{b_0} = T_{\rm rec}\exp\left(-\int_{t_{\rm rec}}^{t_0}\beta dt\right) \ . \tag{2.8}$$

Consequently, the expected temperature anisotropy is

$$\begin{aligned}\frac{\Delta T}{T} &= \frac{T_x - T_z}{T_{\rm rec}} = 1 - \exp\left(\int_{t_{\rm rec}}^{t_0} (\alpha - \beta) dt\right) \\ &\approx \int_{t_{\rm rec}}^{t_0} (\beta - \alpha) dt = -\frac{1}{2}\int_{t_{\rm rec}}^{t_0} \sigma d \ln t \ . \end{aligned} \tag{2.9}$$

By using this expression, Zeldovich and Novikov estimated that a cosmological magnetic field having today the strength of $10^{-9} \div 10^{-10}$ Gauss would produce a temperature anisotropy $\delta T/T \lesssim 10^{-6}$.

The previous analysis has been recently updated by Barrow, Ferreira and Silk [55]. In that work the authors derived an upper limit on the strength of a homogeneous magnetic field at the recombination time on the basis of the 4-year Cosmic Background Explorer (COBE) microwave background isotropy measurements [56]. As it is well know COBE detected quadrupole anisotropies at a level $\delta T/T \sim 10^{-5}$ at an angular scale of few degrees. By performing a suitable statistical average of the data and assuming that the field keeps frozen-in since the recombination till today, Barrow at al. obtained the limit

$$B(t_0) < 3.5 \times 10^{-9} f^{1/2} (\Omega_0 h_{50}^2)^{1/2} \ {\rm G} \ . \tag{2.10}$$

In the above f is a $O(1)$ shape factor accounting for possible non-Gaussian characteristics of the COBE data set.

From this results we see that COBE data are not incompatible with a primordial origin of the galactic magnetic field even without invoking a dynamo amplification.

2.2 The effect on the acoustic peaks

We will now focus our attention on possible effects of primordial magnetic fields on small angular scales. That is, temperature, as well polarization, anisotropies of the CMBR. By small angular scale ($< 1^o$) we mean angles which correspond to a distance smaller than the Hubble horizon radius at the last scattering surface. Therefore, what we are concerning about here are anisotropies that are produced by causal physical mechanisms which are not related to the large scale structure of the space-time.

Primordial density fluctuations, which are necessary to explain the observed structures in the Universe, give rise to acoustic oscillations of the primordial plasma when they enter the horizon some time before the last scattering. The oscillations distort the primordial spectrum of anisotropies by the following primary effects [57]: *a*) they produce temperature fluctuations in the plasma, *b*) they induce a velocity Doppler shift of photons, *c*) they give rise to a gravitational Doppler shift of photons when they climb-out or fall-in the gravitational potential well produced by the density fluctuations (Sachs-Wolfe effect).

In the linear regime, acoustic plasma oscillations are well described by standard fluid-dynamics (continuity + Euler equations) and Newtonian gravity (Poisson's equation). In the presence of a magnetic field the nature of plasma oscillations can be radically modified as Magneto-Hydro-Dynamics (MHD) has to be taken into account.

To be pedagogical, we will first consider a single component plasma and neglect any dissipative effect, due for example to a finite viscosity and heat conductivity. We will also assume that the magnetic field is homogeneous on scales larger than the plasma oscillations wavelength. This choice allows us to treat the background magnetic field $\mathbf{B}_0$ as a uniform field in our equations (in the following symbols with the 0 subscript stand for background quantities whereas the subscript 1 is used for perturbations). Within these assumptions the linearized equations of MHD in comoving coordinates are [58] [1]:

$$\dot{\delta} + \frac{\nabla \cdot \mathbf{v_1}}{a} = 0 \; , \tag{2.11}$$

where a is the scale factor.

$$\dot{\mathbf{v}}_1 + \frac{\dot{a}}{a}\mathbf{v}_1 + \frac{c_S^2}{a}\nabla\delta + \frac{\nabla\phi_1}{a} + \frac{\hat{\mathbf{B}}_0 \times \left(\dot{\mathbf{v}}_1 \times \hat{\mathbf{B}}_0\right)}{4\pi a^4} + \frac{\hat{\mathbf{B}}_0 \times \left(\nabla \times \hat{\mathbf{B}}_1\right)}{4\pi\rho_0 a^5} = 0 \; , \tag{2.12}$$

$$\partial_t \hat{\mathbf{B}}_1 = \frac{\nabla \times \left(\mathbf{v_1} \times \hat{\mathbf{B}}_\mathbf{0}\right)}{a} \; , \tag{2.13}$$

$$\nabla^2\phi_1 = 4\pi G\rho_0 \left(\delta + \frac{\hat{\mathbf{B}}_0 \cdot \hat{\mathbf{B}}_1}{4\pi\rho_0 a^4}\right) \tag{2.14}$$

and

$$\nabla \cdot \hat{\mathbf{B}}_1 = 0 \; , \tag{2.15}$$

[1] Similar equations were derived by Wasserman [25] to the purpose to study the possible effect of primordial magnetic fields on galaxy formation.

where $\hat{\mathbf{B}} \equiv \mathbf{B}a^2$ and $\delta = \dfrac{\rho_1}{\rho_0}$, ϕ_1 and v_1 are small perturbations on the background density, gravitational potential and velocity respectively. c_S is the sound velocity. Neglecting its direct gravitational influence, the magnetic field couples to fluid dynamics only through the last two terms in Eq.(2.12). The first of these terms is due to the displacement current contribution to $\nabla \times B$, whereas the latter account for the magnetic force of the current density. The displacement current term can be neglected provided that

$$v_A \equiv \frac{B_0}{\sqrt{4\pi(\rho + p)}} \ll c_S \ , \tag{2.16}$$

where v_A is the, so called, Alfvén velocity.

Let us now discuss the basic properties of the solutions of these equations, ignoring for the moment the expansion of the Universe. In the absence of the magnetic field there are only ordinary sound waves involving density fluctuations and longitudinal velocity fluctuations (i.e. along the wave vector). By breaking the rotational invariance, the presence of a magnetic field allows new kind of solutions that we list below (useful references on this subject are [59, 60]).

1. *Fast magnetosonic waves*
   In the limit of small magnetic fields these waves become the ordinary sound waves. Their velocity, c_+, is given by

   $$c_+^2 \sim c_S^2 + v_A^2 \sin^2\theta \ , \tag{2.17}$$

   where θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$. Fast magnetosonic waves involve fluctuations in the velocity, density, magnetic field and gravitational field. The velocity and density fluctuations are out-of-phase by $\pi/2$. Eq. (2.17) is valid for $v_A << c_S$. For such fields the wave is approximatively longitudinal.

2. *Slow magnetosonic waves*
   Like the fast waves, the slow waves involve both density and velocity fluctuations. The velocity is however fluctuating both longitudinally and transversely even for small fields. The velocity of the slow waves is approximatively

   $$c_-^2 \sim v_A^2 \cos^2\theta \ . \tag{2.18}$$

3. *Alfvén waves*
For this kind of waves $\mathbf{B}_1$ and $\mathbf{v}_1$ lie in a plane perpendicular to the plane through $\mathbf{k}$ and $\mathbf{B}_0$. In contrast to the magnetosonic waves, the Alfvén waves are purely rotational, thus they involve no density fluctuations. Alfvén waves are linearly polarized. Their velocity of propagation is
$$c_A^2 = v_A^2 \cos^2\theta \ . \tag{2.19}$$

Detailed treatments of the evolution of MHD modes in the matter dominated and radiation dominated eras of the Universe can be found in Refs.[61, 62].

The possible effects of MHD waves on the temperature anisotropies of the CMBR has been first investigated by Adams et al. [58] In the simplest case of magnetosonic waves, they found that the linearized equations of fluctuations in the Fourier space are
$$\dot{\delta}_b + V_b - 3\dot{\phi} = 0, \tag{2.20}$$
$$\dot{V}_b + \frac{\dot{a}}{a}V_b - c_b^2 k^2 \delta_b + k^2\psi + \frac{a n_e \sigma_T (V_b - V_\gamma)}{R} - \frac{1}{4\pi\hat{\rho}_b a}\mathbf{k}\cdot\left(\hat{\mathbf{B}}_0 \times (\mathbf{k}\times\hat{\mathbf{B}}_1)\right) = 0 \ , \tag{2.21}$$
for the baryon component of the plasma and
$$\dot{\delta}_\gamma + \frac{4}{3}V_\gamma - 4\dot{\phi} = 0 \tag{2.22}$$
$$\dot{V}_\gamma - k^2(\frac{1}{4}\delta_\gamma - \sigma_\gamma) - k^2\psi - a n_e \sigma_T (V_b - V_\gamma) = 0 \ , \tag{2.23}$$
for the photon component. In the above $V = i\mathbf{k}\cdot\mathbf{v}$, $R = \dfrac{p_b + \rho_b}{p_\gamma + \rho_\gamma} = \dfrac{3\rho_b}{4\rho_\gamma}$ and c_b is the baryon sound velocity in the absence of interactions with the photon gas. As it is evident from the previous equations, the coupling between the baryon and the photons fluids is supplied by Thomson scattering with cross section σ_T.

In the tight coupling limit ($V_b \sim V_\gamma$) the photons provide the baryon fluid with a pressure term and a non-zero sound velocity. The magnetic field, through the last term in Eq.(2.21), gives rise to an additional contribution to the effective baryon sound velocity. In the case of longitudinal waves this amounts to the change
$$c_b^2 \to c_b^2 + v_A^2 \sin^2\theta \ . \tag{2.24}$$
In other words, the effect of the field can be somewhat mimicked by a variation of the baryon density. A complication arises due to the fact that the

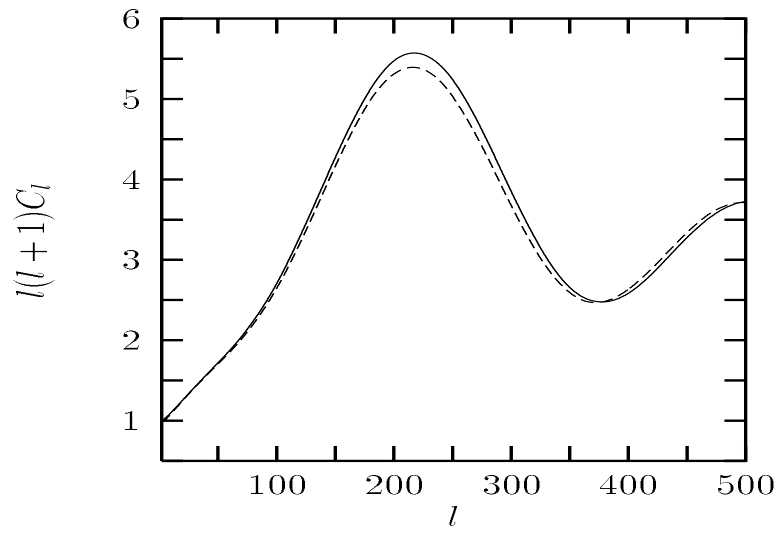

Figure 2.1: The effect of a cosmic magnetic field on the multipole moments. The solid line shows the prediction of a standard CDM cosmology ($\Omega = 1$,$h = 0.5$, $\Omega_{\rm B} = 0.05$) with an $n = 1$ primordial spectrum of adiabatic fluctuations. The dashed line shows the effect of adding a magnetic field equivalent to 2×10^{-7} Gauss today. From Ref.[58]

velocity of the fast waves depends on the angle between the wave-vector and the magnetic field. As we mentioned previously, we are assuming that the magnetic field direction changes on scales larger than the scale of the fluctuation. Different patches of the sky might therefore show different fluctuation spectra depending on this angle.

The authors of Ref.[58] performed an all-sky average summing also over the angle between the field and the line-of-sight. The effect on the CMBR temperature power spectrum was determined by a straightforward modification of the CMBFAST [63] numerical code. From the Fig.2.1 the reader can see the effect of a field $B_0 = 2 \times 10^{-7}$ G on the first acoustic peak. The amplitude of the peak is reduced with respect to the free field case. This is a consequence of the magnetic pressure which opposes the in-fall of the photon-baryon fluid in the potential well of the fluctuation. Although this is not clearly visible from the figure, the variation of the sound velocity, hence of the sound horizon, should also produce a displacement of the acoustic peaks. The combination of these two effects may help to disentangle the signature of the magnetic field from other cosmological effects (for a comprehensive review see [64]) once more precise observations of the CMBR power spectrum will be available. Adams at al. derived an estimate of the sensitivity to B which MAP [66] and PLANCK [67] satellites observations should allow to reach by translating the predicted sensitivity of these observations to Ω_b. They found that a magnetic field with strength today $B_0 > 5 \times 10^{-8}$ G should be detectable.

It is interesting to observe that a magnetic field cannot lower the ratio of the first to second acoustic peak as showed by recent observations [65].

Alfvén waves may also leave a signature on the CMBR anisotropies. There are at least three main reasons which make this kind of wave of considerable interest. The first is that Alfvén waves should leave a quite peculiar imprint on the CMBR power spectrum. In fact, as we discussed in the above, these waves do not involve fluctuations in the density of the photon-baryon fluid. Rather, they consist only of oscillations of the fluid velocity and of the magnetic field. Indeed, by assuming that the wavelength is smaller than the Hubble radius and that relativistic effects are negligible, the equations describing Alfvén waves are [58]

$$\delta_b = \delta_\gamma = 0 \ , \tag{2.25}$$

$$\dot{\mathbf{v}}_b + \frac{\dot{a}}{a}\mathbf{v}_b + \frac{a n_e \sigma_T (\mathbf{v}_b - \mathbf{v}_\gamma)}{R} - i\frac{(\mathbf{k}\cdot\hat{\mathbf{B}}_\mathbf{0})}{4\pi\hat{\rho}_b a}\hat{\mathbf{B}}_1 = 0 \ , \tag{2.26}$$

$$\dot{\mathbf{v}}_\gamma - a n_e \sigma_T (\mathbf{v}_b - \mathbf{v}_\gamma) = 0 \tag{2.27}$$

$$\phi = 0 \ . \tag{2.28}$$

Since the gravitational Doppler shift (Sachs-Wolfe effect) is absent in this case, the cancellation against the velocity Doppler shift which occurs for the acoustic modes [57] does not take place for the Alfvén waves. This could provide a more clear signature of the presence of magnetic fields at the last scattering surface [58].

The second reason why Alfvén waves are so interesting in this contest is that they are vector (or rotational) perturbations. As a consequence they are well suited to probe peculiar initial condition such as those that might be generated from primordial phase-transitions. It is remarkable that whereas vector perturbations are suppressed by universe expansion and cannot arise from small deviations from the isotropic Friedmann Universe for $t \to 0$ [53], this is not true in the presence of a cosmic magnetic field [2].

The third reason of our interest for Alfvèn waves is that for this kind of waves the effect of dissipation is less serious than what it is for sound and fast magnetosonic waves. This issue will be touched upon in the next section.

A detailed study of the possible effects of Alfvén waves on the CMBR anisotropies has been independently performed by Subramanian and Barrow [69] and Durrer at al.[70] who reached similar results. We summarize here the main points of the derivation as given in Ref.[70].

In general, vector perturbations of the metric have the form

$$(h_{\mu\nu}) = \begin{pmatrix} 0 & B_i \\ B_j & H_{i,j} + H_{j,i} \end{pmatrix} , \tag{2.29}$$

where $\mathbf{B}$ and $\mathbf{H}$ are divergence-free, 3d vector fields supposed to vanish at infinity. Two gauge-invariant quantities [71] are conveniently introduced by the authors of Ref.[70]:

$$\sigma = \dot{\mathbf{H}} - \mathbf{B} \quad \text{and} \quad \mathbf{\Omega} = \mathbf{v} - \mathbf{B} . \tag{2.30}$$

which represents the vector contribution to the perturbation of the extrinsic curvature and the vorticity. In the absence of the magnetic field, and assuming a perfect fluid equation of state, the vorticity equation of motion is

$$\dot{\mathbf{\Omega}} + (1 - 3c_s^2)\frac{\dot{a}}{a}\mathbf{\Omega} = 0 . \tag{2.31}$$

In the radiation dominated era the solution of this equation is $\Omega = \text{const.}$ which clearly does not describe waves and, as we mentioned, is incompatible with an isotropic universe when $t \to 0$. In the presence of the magnetic field, Durrer et al. found

$$\ddot{\mathbf{\Omega}} = \frac{(\mathbf{B}_0 \cdot \mathbf{k})^2}{4\pi(\rho_r + p_r)}\mathbf{\Omega} \quad \text{and} \tag{2.32}$$

$$\dot{\mathbf{\Omega}} = \frac{i\mathbf{B}_0 \cdot \mathbf{k}}{4\pi(\rho_r + p_r)}\mathbf{B}_1 . \tag{2.33}$$

[2] Collisionless matter, like e.g. gravitons after the Planck era, may however support nonzero vorticity even with initial conditions compatible with an isotropic universe [68].

These equations describe Alfvén waves propagating at the velocity $v_A(\mathbf{e}\cdot\hat{\mathbf{k}})$, where v_A is the Alfvén velocity and $\mathbf{e}$ is the unit vector in the direction of the magnetic field [3]. In this case some amount of initial vorticity is allowed which is connected to the amplitude of the magnetic field perturbation $\mathbf{B}_1$

$$|\mathbf{\Omega_0}| = (v_A/B_0)|\mathbf{B}_1| \ . \tag{2.34}$$

The general form of the CMBR temperature anisotropy produced by vector perturbations is

$$\left(\frac{\Delta T}{T}\right)^{(\mathrm{vec})} = -\ \mathbf{V}\cdot\mathbf{n}|_{t_{dec}}^{t_0} + \int_{t_{dec}}^{t_0} \dot{\sigma}\cdot\mathbf{n}d\lambda \ , \tag{2.35}$$

where $\mathbf{V} = \mathbf{\Omega} - \sigma$ is a gauge-invariant generalization of the velocity field. We see from the previous equation that besides the Doppler effect Alfvèn waves gives rise to an integrated Sachs-Wolfe term. However, since the geometric perturbation σ is decaying with time, the integrated term is dominated by its lower boundary and just cancels σ in $\mathbf{V}$. Neglecting a possible dipole contribution from vector perturbations today, Durrer at al. obtained

$$\frac{\delta T}{T}(\mathbf{n},\mathbf{k}) \simeq \mathbf{n}\cdot\mathbf{\Omega}(\mathbf{k},t_{dec}) = \mathbf{n}\cdot\mathbf{\Omega}_0 \sin\left(v_A k t_{dec}(\mathbf{e}\cdot\hat{\mathbf{k}})\right) \ . \tag{2.36}$$

As predicted in Ref.[58], Alfvén waves produce Doppler peaks with a periodicity which is determined by the Alfvén velocity. Since, for reasonable values of the magnetic field strength, $v_A \ll 1$ this peaks will be quite difficult to detect.

Durrer et al. argued that Alfvén waves may leave a phenomenologically more interesting signature on the statistical properties of the CMBR anisotropies. In the absence of the magnetic field all the relevant information is encoded in the C_ℓ's coefficients defined by

$$\left\langle\frac{\delta T}{T}(\mathbf{n})\frac{\delta T}{T}(\mathbf{n}')\right\rangle\Bigg|_{(\mathbf{n}\cdot\mathbf{n}'=\mu)} = \frac{1}{4\pi}\sum_\ell (2\ell+1)C_\ell P_\ell(\mu) \ , \tag{2.37}$$

where $\mu \equiv \mathbf{n}\cdot\mathbf{n}'$. By introducing the usual spherical harmonics decomposition

$$\frac{\delta T}{T}(\mathbf{n}) = \sum_{\ell,m} a_{\ell m} Y_{\ell m}(\mathbf{n}), \tag{2.38}$$

[3]Differently form the authors of Ref.[58], Durrer at al. assumed a homogeneous background magnetic field . This however is not a necessary condition for the validity of the present considerations.

the C_ℓ's are just

$$C_\ell = \langle a_{\ell m} a^*_{\ell m} \rangle \ . \tag{2.39}$$

Because of its spin-1 nature, the vorticity vector field induces transitions $\ell \to \ell \pm 1$ hence a correlation between the multipole amplitudes $a_{\ell+1,m}$ and $a_{\ell-1,m}$. This new kind of correlation is encoded in the coefficients

$$D_\ell(m) = \langle a_{\ell-1,m} a^*_{\ell+1,m} \rangle = \langle a_{\ell+1,m} a^*_{\ell-1,m} \rangle \ . \tag{2.40}$$

Durrer at al. [70] determined the form of the C_ℓ and D_ℓ coefficients for the case of a homogeneous background magnetic field in the range $-7 < n < -1$, where n determine the vorticity power spectrum according to

$$\langle \Omega_{0i}(\mathbf{k}) \Omega_{0j}(\mathbf{k}) \rangle \ = \ (\delta_{ij} - \hat{k}_i \hat{k}_j) A(|\mathbf{k}|) \tag{2.41}$$

$$A(k) = \ A_0 \ \frac{k^n}{k_0^{(n+3)}}, \qquad k < k_0 \ . \tag{2.42}$$

On the basis of these considerations they found that 4-year COBE data allow to obtain a limit on the magnetic field amplitude in the range $-7 < n < -3$ on the order of $(2-7) \times 10^{-9}$Gauss.

2.3 Dissipative effects on the MHD modes

In the previous section we neglected any dissipative effect which may possibly affect the evolution of the MHD modes. However, similar to the damping of baryon-photon sound waves by photon shear viscosity and heat conductivity, damping of MHD perturbations may also occur. This issue was studied in detail by Jedamzik, Katalinić and Olinto [72] who first determined the damping rates of fast and slow magnetosonic waves as well as of Alfvén waves. Furthermore, it was shown in Refs.[72, 73] that dissipation of MHD modes produce an effective damping of inhomogeneous magnetic fields. The dissipation process goes as follows. A spatially tangled magnetic field produces Lorentz forces which accelerate the plasma and set up oscillations. Since the radiation-baryon pressure is much larger than the magnetic pressure, as long as the photon mean-free-path is smaller than the scale of the magnetic tangle, the motions can be considered as being largely incompressible. In this situation mainly Alfvén waves, which do not involve density fluctuations, are excited. In the absence of dissipation, this process will continue until, for all scales λ with magnetic field relaxation time $\tau \sim \lambda/v_A$ shorter than the Hubble time t_H, an approximate equipartition between magnetic and kinetic energies is produced. If the fluid is non-ideal, however, shear viscosity will induce dissipation of kinetic energy, hence also of magnetic energy, into heat. In this case dissipation will end only when the magnetic field reaches a force-free state.

In the absence of magnetic fields it is known that in the diffusive regime (i.e. when the perturbation wavelength is much larger than the mean free path of photon or neutrinos) acoustic density fluctuations are effectively damped because of the finite viscosity and heat conductivity (Silk damping [74]). At recombination time, dissipation occurs for modes smaller than the approximate photon diffusion length, $d_\gamma \sim (l_\gamma t_H)^{1/2}$, where l_γ is photon mean free path. The dissipation of fast magnetosonic waves proceeds in a quite similar way. Indeed, it was showed in Ref.[72] that the dissipation length scale of these kind of waves coincide with the Silk damping scale. More interesting is the result found in Refs.[72, 73] which shows that damping of Alfvén and slow magnetosonic waves is significantly different from damping of sound and fast magnetosonic waves. The reason for such a different behavior is that, for a small background magnetic field $v_A \ll 1$ so that the oscillation frequency of an Alfvén mode ($v_A k/a$) is much smaller than the oscillation frequency of a fast magnetosonic mode with the same wavelength ($v_{\rm sound} k/a$). While all magnetosonic modes of interest satisfy the condition for damping in the oscillatory regime ($v_{\rm sound} \ll l_\gamma k/a$), an Alfvén mode can become *overdamped* when the photon (or neutrino) mean-free-path becomes large enough for dissipative effects to overcome the oscillations ($v_A \cos\theta \simeq l_\gamma(T) k/a$, where θ is the angle between the background magnetic field and the wave vector). Because of the strong viscosity, that prevent fluid acceleration by the magnetic forces, damping is quite inefficient for non-oscillating overdamped Alfvén modes with

$$\lambda \le \lambda_{od} \simeq \frac{2\pi l_\gamma(T)}{v_A \cos\theta} \ . \tag{2.43}$$

As a result, the damping scale of overdamped Alfvén modes at the end of the diffusion regime is smaller than the damping scale of sound and fast magnetosonic modes (Silk damping scale) by a factor which depends on the strength of the background magnetic field and the θ angle, $L_A \sim v_A \cos\theta d_\gamma$.

From the previous considerations it follows that the results discussed in the previous section hold only under the assumption that the magnetic field coherence length is not much smaller than the comoving Silk damping scale ($L_S \sim 10$ Mpc), in the case of fast magnetosonic waves, and not smaller than

L_A for Alfvén waves.

Some other interesting work has been recently done by Jedamzik, Katalinić and Olinto [75] concerning the effects of dissipation of small-scale magnetic fields on the CMBR. The main idea developed in the paper by Jedamzik et al. is that the dissipation of tangled magnetic fields before the recombination epoch should give rise to a nonthermal injection of energy into the heat-bath which may distort the thermal spectrum of CMBR. It was showed by the authors of Ref.[75] that once photon equilibration has occurred, mainly via photon-electron scattering and double-Compton scattering, the resultant distribution should be of Bose-Einstein type with a non-vanishing chemical potential. The evolution of the chemical potential distortions at large frequencies may be well approximated by [76]

$$\frac{d\mu}{dt} = -\frac{\mu}{t_{DC}(z)} + 1.4\frac{Q_B}{\rho_\gamma} , \tag{2.44}$$

where, in our case, $Q_B = d\rho_B/dt$ is the dissipation rate of the magnetic field and $t_{DC} = 2.06 \times 10^{33}$ s $(\Omega_b h^2)^{-1} z^{-9/2}$ is a characteristic time scale for double-Compton scattering. Jedamzik et al. assumed a statistically isotropic magnetic field configuration with the following power spectrum

$$|\tilde{b}_k|^2 = B_0^2 \left(\frac{k}{k_N}\right)^n \frac{(n+3)}{4\pi} \qquad \text{for} \qquad k < k_N \tag{2.45}$$

and zero otherwise, normalized such that $\langle \tilde{b}^2 \rangle = B_0^2$. The energy dissipation rate was determined by substituting this spectrum in the following Fourier integral

$$Q_B = \frac{1}{8\pi k_N^3} \int d^3k \, \frac{d|\tilde{b}_k|^2}{dt} = \frac{1}{8\pi k_N^3} \int d^3k \, |\tilde{b}_k|^2 \; (2\mathrm{Im}\,\omega) \; \exp\Big(-2\int \mathrm{Im}\,\omega dt\Big) \; , \tag{2.46}$$

together with the mode frequencies for Alfvén and slow magnetosonic waves determined in Ref.([72])

$$\omega_{\rm osc}^{\rm SM,A} = v_A \cos\theta \Big(\frac{k}{a}\Big) + \frac{3}{2} i \frac{\eta'}{(1+R)} \Big(\frac{k}{a}\Big)^2 , \tag{2.47}$$

where $3(\rho_\gamma + p_\gamma)\eta' = \eta$, and η is the shear viscosity. For $k_N \gg k_D^0 z_\mu^{3/2}$, where $k_D^0 = (15 n_e^0 \sigma_{Th}/2.39 \times 10^19 \text{ s})^{1/2}$, an analytic solution of Eq.(2.44) was then found to be

$$\mu = K \frac{B_0^2}{8\pi\rho_\gamma^0}\left(\frac{k_D^0}{k_N} z_\mu^{3/2}\right)^{(n+3)} . \tag{2.48}$$

In the above K is a numerical factor of order 1, the precise value depending on the spectral index n and z_μ is the characteristic redshift for "freze-out" from double-Compton scattering. This redshift equals $z_\mu = 2.5 \times 10^6$ for typical values $\Omega_b h^2 = 0.0125$, and $Y_p = 0.24$. The scale $k_D^0 z_\mu^{3/2}$ has a simple interpretation. It is the scale which at redshift z_μ is damped by one e-fold. For the above values of $\Omega_b h^2$ and Y_p the corresponding comoving wavelength is $\lambda_D = (2\pi)/(k_D^0 z_\mu^{3/2}) = 395$pc.

The present upper limit on chemical potential distortion of the CMBR come from the COBE/FIRAS data: $|\mu| < 9 \times 10^{-5}$ at 95% confidence level [77]. Comparing this limit with the prediction of Eq.(2.48) it follows that primordial magnetic fields of strength $\gtrsim 3 \times 10^{-8}$ G, and comoving coherence length ≈ 400 pc are probably excluded. On slightly larger scales, dissipation of spatially tangled magnetic fields may give to a different kind of CMBR distortion which may be described by a superposition of blackbodies of different temperature, i.e. a Compton y distortion [78]. The absence of this kind of distortions in the observed CMBR thermal spectrum disallow magnetic fields of $\gtrsim 3 \times 10^{-8}$Gauss on scales ~ 0.6 Mpc.

2.4 Effects on the CMBR polarization

Thomson scattering is a natural polarizing mechanism for the CMBR. It is enough if the photon distribution function seen by the electrons has a quadrupole anisotropy to obtain polarization. At early times, the tight coupling between the photons and the electron-baryon fluid prevents the development of any photon anisotropy in the baryon's rest frame, hence the polarization vanishes. As decoupling proceeds, the photons begin to free-stream and temperature quadrupole anisotropies can source a space dependent polarization. For this reason temperature and polarization anisotropies are expected to be correlated (for a comprehensive review on the subject see [79]).

The expected polarization anisotropy is not large, perhaps about 10^{-6}. Currently the best polarization limit comes from the Saskatoon experiment [80], with a 95% confidence level upper limit of 25 μK at angular scales of about a degree, corresponding to 9×10^{-6} of the mean temperature. Future balloons and satellites observations, like e.g. the PLANCK [67] mission to be launched in 2007, are expected to have a good chance to measure the CMBR polarization power spectrum.

Kosowsky and Loeb [81] first observed that the possible presence of magnetic fields at the decoupling time may induce a sizeable Faraday rotation in the CMBR. Since the rotation angle depend on the wavelength, it is possible to estimate this effect by comparing the polarization vector on a given direction at two different frequencies. The basic formula is [15]

$$\phi = \frac{e^3 n_e x_e \mathbf{B} \cdot \hat{\mathbf{q}} \lambda^2 L}{8\pi^2 m^2 c^2} \tag{2.49}$$

where ϕ is the amount by which the plane of polarization of linearly polarized radiation has been rotated, after traversing a distance L in a homogeneous magnetic field B in a direction $\hat{\mathbf{q}}$. x_e is the ionized fraction of the total electron density n_e and m is the electron mass. Finally λ is the radiation wavelength.

Although the magnetic field strength is expected to be larger at early times, the induced Faraday rotation depends also on the free electron density (see Eq.(2.49)) which drops to negligible values as recombination ends. Therefore, rotation is generated during the brief period of time when the free electron density has dropped enough to end the tight coupling but not so much that Faraday rotation ceases. A detailed computation requires the solution of the radiative transport equations in comoving coordinates [81]

$$\begin{aligned}
\dot{\Delta}_T + ik\mu(\Delta_T + \Psi) &= -\dot{\Phi} - \dot{\kappa}[\Delta_T - \Delta_T(0) - \mu V_b + \frac{1}{2}P_2(\mu)S_P] && (2.50)\\
\dot{\Delta}_Q ik\mu\Delta_Q &= -\dot{\kappa}[\Delta_Q - \frac{1}{2}(1 - P_2(\mu)S_P + 2\omega_B\Delta_U] && (2.51)\\
\dot{\Delta}_U + ik\mu\Delta_U &= -\dot{\kappa}\Delta_U - 2\omega_B\Delta_Q \ . && (2.52)
\end{aligned}$$

In the above Δ_T, Δ_Q and Δ_U respectively represent the fluctuations of temperature and of the of the Stokes parameters Q and U [15]. The linear polarization is $\Delta_P = \sqrt{\Delta_Q^2 + \Delta_U^2}$. The numerical subscripts on the radiation brightnesses Δ_X indicate moments defined by an expansion of the directional dependence in Legendre polynomials $P_\ell(\mu)$:

$$\Delta_{X\,\ell}(k) \equiv \frac{1}{2}\int_{-1}^{1} d\mu\, P_\ell(\mu)\Delta_X(k,\mu). \tag{2.53}$$

Therefore, the subscripts 0, 1, 2 label respectively monopole, dipole and quadrupole moments. $\dot{\kappa} = x_e n_e \sigma_T \frac{\dot{a}}{a}$ is the differential optical depth and the quantities V_b and R have been defined in the previous section. Derivatives are respect to conformal time. Finally, ω_B is the Faraday rotation rate

$$\omega_B = \frac{d\phi}{d\tau} = \frac{e^3 n_e x_e \mathbf{B} \cdot \hat{\mathbf{q}}}{8\pi^2 m^2 \nu^2} \frac{a}{a_0} \ . \tag{2.54}$$

From Eqs.(2.51,2.52) wee see that Faraday rotation mixes Q and U Stokes parameters. The polarization brightness Δ_Q is induced by the function $S_P = -\Delta_T(2) - \Delta_Q(2) + \Delta_Q(0)$ and Δ_U is generated as Δ_Q and Δ_U are rotated into each other. In the absence of magnetic fields, Δ_U retains its tight-coupling value of zero. The set of Eqs.(2.50-2.52) is not easily solved. A convenient approximation is the tight coupling approximation which is an expansion in powers of $k\tau_C$ with $\tau_C = \dot{\kappa}^{-1}$. This parameter measures the average conformal time between collisions. At decoupling the photon mean free path grows rapidly and the approximation breaksdown, except for long wavelength as measured with respect to the thickness of the last scattering surface. For these frequencies the approximation is still accurate.

Kosowsky and Loeb assumed a uniform magnetic field on the scale of the width of the last scattering surface, a comoving scale of about 5 Mpc. This assumption is natural if the coherent magnetic field observed in galaxies comes from a primordial origin, since galaxies were assembled from a comoving scale of a few Mpc. The mean results was obtained by averaging over the entire sky. Therefore the equations still depend only on k and $\mu = \cos(\hat{\mathbf{k}} \cdot \hat{\mathbf{q}})$ and not on the line-of-sight vector $\hat{\mathbf{q}}$ and the perturbation wave-vector $\mathbf{k}$ separately. The evolution of the polarization brightnesses, for given values of k and μ is represented in Fig.2.2 as a function of the redshift.

By following the approach described in the above Kosowsky and Loeb estimated the polarization angle produced by a magnetic field on CMB photons with frequency $\nu_1 < \nu_2$ to be

$$\langle \varphi_{12}^2 \rangle^{1/2} = 1.1^\circ \left(1 - \frac{\nu_1^2}{\nu_2^2}\right) \left(\frac{B_0}{10^{-9}\,\mathrm{G}}\right) \left(\frac{30\,\mathrm{GHz}}{\nu_1}\right)^2 \ . \tag{2.55}$$

A 10 % correction may apply to this expression to account for the effects $\Omega_b h^2$ and $\Omega_0 h^2$ (in the range $\Omega_b h^2 > 0.007$ and $\Omega_0 h^2 < 0.3$).

For a primordial field of $B_0 \sim 10^{-9}$ G which could result in the observed galactic field without dynamo amplification, one can therefore expect

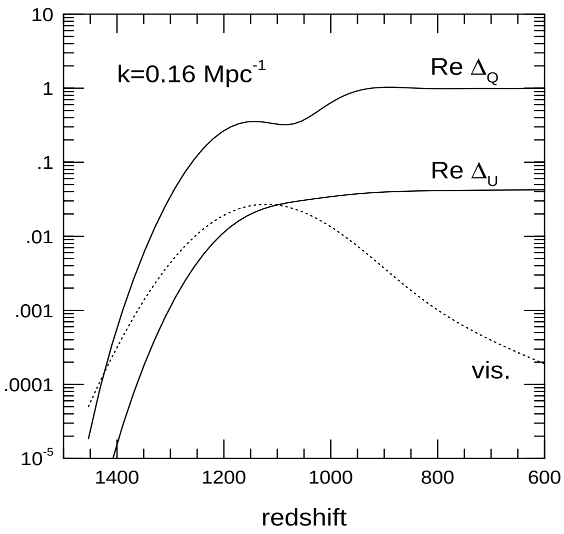

Figure 2.2: The evolution of the polarization brightnesses, for $k = 0.16\,\mathrm{Mpc}^{-1}$ and $\mu = 0.5$ (in arbitrary units). Also plotted as a dotted line is the differential visibility function $\dot{\tau}e^{-\tau}$ in units of Mpc^{-1}. From Ref.[81].

a rotation measure of order 1.6 deg cm^{-2} = 280 rad m^{-2}. This rotation is considerable by astrophysical standards and could in principle be measured.

We noticed at the beginning of this section that temperature and polarization anisotropies of the CMBR are generally expected to be correlated. The statistical properties of such correlation may be affected by the presence of a magnetic field at the decoupling time in a peculiar way. In fact, it was showed by Scannapieco and Ferreira [82] that such a field may induce an observable parity odd cross correlation between polarization and temperature anisotropies. Any polarization pattern on the sky can be separated into "electric" (E) and "magnetic" (B) components. The nomenclature reflects the global parity property. Like multipole radiation, the harmonics of an E-mode have $(-1)^\ell$ parity on the sphere, whereas those of a B-mode have $(-1)^{\ell+1}$ parity. Indeed, given a measurement of the Stokes parameters Q and

U, this data can be decomposed into a sum over spin ± 2 spherical harmonics

$$(Q \pm iU)(\mathbf{n}) = \sum_{\ell m} a_{\ell m}^{\pm 2}\ {}_{\pm 2}Y_{\ell m}(\mathbf{n})\ . \tag{2.56}$$

Under parity inversion, ${}_sY_{\ell m} \to (-1)^\ell\ {}_sY_{\ell m}$ so that ${}_2Y_{\ell m} \pm {}_{-2}Y_{\ell m}$ are parity eigenstates. It is then convenient to define the coefficients

$$a_{\ell m}^E \equiv -\frac{1}{2}(a_{\ell m}^2 + a_{\ell m}^{-2}) \quad a_{\ell m}^B \equiv \frac{i}{2}(a_{\ell m}^2 - a_{\ell m}^{-2}) \tag{2.57}$$

so that the E-mode remains unchanged under parity inversion for even ℓ, whereas the B-mode changes sign.

In an isotropic Universe, cross correlation between the B and E polarizations is forbidden as this would imply parity violation. Magnetic fields, however, are maximally parity violating and therefore they may reveal their presence by producing such a cross correlation, Faraday rotation being the physical process which is responsible for this effect. The authors of Ref.[82] determined the expected cross correlation between temperature and E and B polarization modes. On the basis of such result they concluded that magnetic fields strengths as low as 10^{-9} (present time value obtained assuming adiabatic scaling) could be detectable by the PLANCK satellite mission. It is worthwhile to note that Scannapieco and Ferreira only considered homogeneous magnetic fields. We note, however, that most of their considerations should apply also to the case of magnetic fields with a finite coherence length. In this case measurements taken in different patches of the sky should present different temperature-polarization cross correlation depending on the magnetic field and the line-of-sight direction angle.

The consequences of Faraday rotation may go beyond the effect they produce on the CMBR polarization. Indeed, Harari, Hayward and Zaldarriaga [83] observed that Faraday rotation may also perturb the temperature power spectrum of CMBR. The effect mainly comes as a back-reaction of the radiation depolarization which induces a larger photon diffusion length reducing the viscous damping of temperature anisotropies.

In the absence of the magnetic field ($\omega_B = 0$), to the first order in the tight-coupling approximation one finds

$$\Delta_U = 0, \quad \Delta_Q = \frac{3}{4} S_P sin^2(\theta) \tag{2.58}$$

and

$$S_P = -\frac{5}{2}\Delta_T(2) = \frac{4}{3}ik\tau_C\Delta_T(1) = -\frac{4}{3}\tau_C\dot{\Delta}_0 \tag{2.59}$$

$\Delta_0 = \Delta_T(0) + \Phi$. Obviously, all multipoles with $l > 3$ vanish to this order. Replacing all quantities in terms of Δ_0 one obtains [84]:

$$\ddot{\Delta} + \left(\frac{\dot{R}}{1+R} + \frac{16}{45}\frac{k^2\tau_C}{1+R}\right)\dot{\Delta}_0 + \frac{k^2}{3(1+R)}\Delta_0 = \frac{k^2}{3(1+R)}\left(\Phi - (1+R)\Psi\right) \tag{2.60}$$

that can be interpreted as the equation of a forced oscillator in the presence of damping.

In the presence of the magnetic field $\omega_B \neq 0$. The depolarization depends upon two angles: a) the angle between the magnetic field and wave propagation and b) the angle of the field with the wave vector $\mathbf{k}$. Since we assume that the vector $\mathbf{k}$ is determined by stochastic Gaussian fluctuations, its spectrum will have no preferred direction. Therefore this dependence will average out when integrated. It is also assumed that for evolution purposes, the magnetic field has no component perpendicular to $\mathbf{k}$. This imposed axial symmetry is compatible with the derivation of the above written Boltzmann equations. Under these assumptions Harari et al. found [83]

$$\Delta_U = -F\cos\theta\Delta_Q \qquad \Delta_Q = \frac{3}{4}\frac{S_P\sin^2\theta}{(1+F^2\cos^2\theta)} \tag{2.61}$$

where the coefficient F was defined by

$$F\cos\theta \equiv 2\omega_B\tau_C \tag{2.62}$$

which gives

$$F = \frac{e^3}{4\pi^2 m^2\sigma_T}\frac{B}{\nu^2} \sim 0.7\left(\frac{B_*}{10^{-3}\mathrm{G}}\right)\left(\frac{10\ \mathrm{Ghz}}{\nu_0}\right)^2 . \tag{2.63}$$

Physically, F represents the average Faraday rotation between two photon-electron scattering. Note that assuming perfect conductivity

$$B(t) = B(t_*)\left(\frac{a(t_*)}{a(t)}\right)^2 \tag{2.64}$$

and therefore F is a time independent quantity. Faraday rotation between collisions becomes considerably large at frequencies around and below ν_d. This quantity is implicitly defined by

$$F \equiv \left(\frac{\nu_d}{\nu_0}\right)^2 \tag{2.65}$$

which gives

$$\nu_d \sim 8.4 \text{ Ghz } 9 \left(\frac{B_*}{10^{-3}\text{ G}}\right)^{\frac{1}{2}} \sim 27 \text{ Ghz } \left(\frac{B_*}{10^{-2}\text{ G}}\right)^{\frac{1}{2}} . \tag{2.66}$$

From Eqs.(2.61) and, the definition of S_P given in the first part of this section, one can extract

$$\Delta_{Q_0} = \frac{1}{2} d_0(F) S_P, \qquad \Delta_{Q_2} = -\frac{1}{10} d_2(F) S_P, \tag{2.67}$$

$$\Delta_{T_2} = -S_P(1 - \frac{3}{5}d) \tag{2.68}$$

and, from the equation for Δ_T in the tight coupling,

$$S_P = \frac{4}{3(3-2d)} ik\tau_C \Delta_{T_1} = -\frac{4}{3(3-2d)} \tau_C \dot{\Delta}_0 . \tag{2.69}$$

In the above the coefficients are defined so that $d_i \approx 1 + O(F^2)$ for small F, i.e. small Faraday rotation, while $d_i \to O(1/F)$ as $F \to \infty$ (for the exact definition see Ref.[83]). Equations (2.61,2.68) and (2.69) condense the main effects of a magnetic field upon polarization. When there is no magnetic field $(F = 0, d = 1)$ $\Delta_U = 0$ and $\Delta_Q = -\frac{15}{8}\Delta_{T_2} \sin^2\theta$. A magnetic field generates Δ_U, through Faraday rotation, and reduces Δ_Q. In the limit of very large F (large Faraday rotation between collisions) the polarization vanishes. The quadrupole anisotropy Δ_{T_2} is also reduced by the depolarizing effect of the magnetic field, by a factor $5/6$ in the large F limit, because of the feedback of Δ_Q upon the anisotropy or, in other words, because of the polarization dependence of Thomson scattering. The dipole Δ_{T_1} and monopole Δ_{T_0} are affected by the magnetic field only through its incidence upon the damping mechanism due to photon diffusion for small wavelengths. Indeed, the equation for $\Delta_0 = \Delta_{T_0} + \Phi$, neglecting $O(R^2)$ contributions, now reads

$$\ddot{\Delta}_0 + \left(\frac{\dot{R}}{1+R} + \frac{16}{90}\frac{(5-3d)}{(3-2d)}\frac{k^2\tau_C}{(1+R)}\right)\dot{\Delta}_0 + \frac{k^2}{3(1+R)}\Delta_0 = \frac{k^2}{3(1+R)}\left(\Phi - (1+R)\Psi\right) \tag{2.70}$$

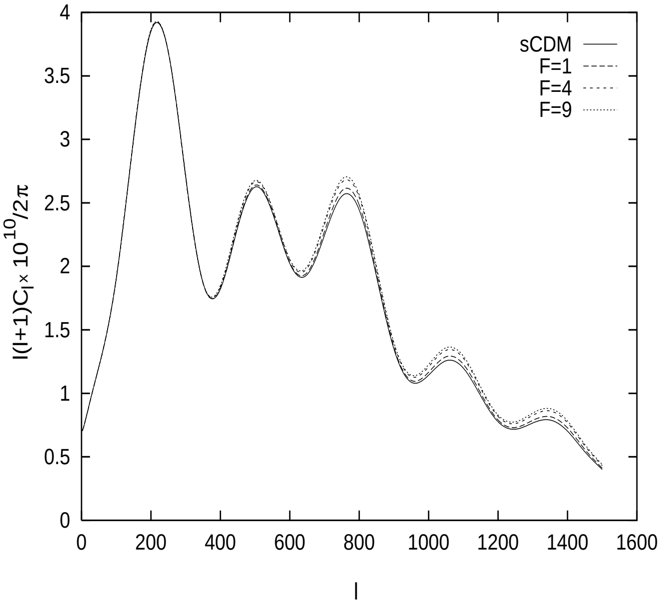

Figure 2.3: Numerical integration for the multipoles of the anisotropy correlation function in a standard CDM model without a primordial magnetic field ($F = 0$), and with $F = 1,\ 4,\ 9$, which correspond to $\nu_0 = \nu_{\rm d},\ \nu_{\rm d}/2,\ \nu_{\rm d}/3$ respectively, with $\nu_{\rm d} \approx 27$ GHz $(B_*/0.01\text{Gauss})^{1/2}$. From Ref.[83].

which is the equation of a damped harmonic oscillator.

The damping of the temperature anisotropies on small angular scales can be determined by solving the radiative transfer equation to second order in the tight-coupling approximation. By assuming solutions of the form

$$\Delta_X(\tau) = \Delta_X e^{i\omega\tau} \tag{2.71}$$

for $X = T$, Q, and U, and similarly for the baryon velocity V_b, Harari et al. [83] found the following solution for Eq.(2.70)

$$\omega = \frac{k}{\sqrt{3(1+R)}} + i\gamma\ , \tag{2.72}$$

where the photon-diffusion damping length-scale is

$$\gamma(d) \equiv \frac{k^2}{k_D^2} = \frac{k^2\tau_C}{6(1+R)}\Big(\frac{8}{15}\frac{(5-3d)}{(3-2d)} + \frac{R^2}{1+R}\Big)\ . \tag{2.73}$$

The damping affects the multipole coefficients of the anisotropy power spectrum which are defined by

$$C_l = (4\pi)^2 \int k^2 dk P(k) |\Delta_{T_l}(k, \tau_0)|^2 \ . \tag{2.74}$$

The average damping factor due to photon diffusion upon the C_l's is given by an integral of $e^{-2\gamma}$ times the visibility function across the last scattering surface [84, 85]. It depends upon cosmological parameters, notably R, and upon the recombination history.

In the Fig.2.3 it is represented the correction to the temperature power spectrum expected for several values of the parameter F. We see from that figure that on small angular scales the effect of the magnetic field is to increase the temperature anisotropies. The magnitude of this effect was estimated to be up to 7.5% in a CDM Universe on small angular scales ($l \approx 1000$) at a level that should be reachable from future CMBR satellite experiments like MAP [66] and PLANCK [67]. The frequency at which the effect should be detectable will, however, depend on the strength and coherence length of the magnetic field at the recombination time. Both experiment should be sensitive to magnetic fields around $B_{z=1000} = 0.1$G or, equivalently, $B_0 = 10^{-7}$G a level that is comparable to be BBN limit (see Chap.3).

Chapter 3

Constraints from the Big Bang Nucleosynthesis

The study of the influence of magnetic fields on the Big Bang Nucleosynthesis (BBN) began with the pioneering works of Matese and O'Connell [86, 87, 88] and Greenstein [89]. It is remarkable that most of the more relevant effects were already pointed-out in those early papers.

In their first paper on the subject Matese and O'Connell [86] showed that in the presence of very strong magnetic fields, $B > B_c \equiv eB/m_e^2 = 4.4 \times 10^{13}$ G (above this field strength quantized magnetic levels, "cyclotron lines", appear), the β decay rate of neutrons is significantly increased. This is mainly a consequence of the periodicity of the electron wave function in the plane normal to the field which turns into an enlarging of the electron available phase-space. Since the magnetic fields required to obtain any sizeable effect cannot be reached in the laboratory in the foreseeable future, Matese and O'Connell addressed their attention to the early Universe. The effects of primordial magnetic fields on the production of ^{4}He during BBN was first considered in Ref.[87]. On the basis of the results obtained in their previous work [86], Matese and O'Connell argued that strong magnetic fields should suppress the ^{4}He relic abundance with respect to the standard case. Briefly, their argument was the following. Since, after the neutron to proton ratio has been frozen, it takes some time for neutrons to be bounded into composite nuclei, a faster neutron decay due to the magnetic field implies smaller relic abundances of ^{4}He and of the heavier elements.

In Ref. [87] two other possible effects of a magnetic field on BBN were shortly considered. The first of these effects consists in the variation that a strong magnetic field induces on the energy density of the electron-positron gas. This effect is a consequence of the grow of the electron and positron phase-space in the presence of over-critical ($B > B_c$) magnetic fields. Below we shall show how such an effect may have relevant consequences on the BBN through its action on the expansion rate of the Universe and the entropy transfer from the e^+e^- gas to the photons. The second effect touched by Matese and O'Connell concerns the influence of a uniform magnetic field on the Universe geometry and its consequences on the BBN [1]. Matese and O'Connell analysis of these two effects was only qualitative and, as far as we know, no further work was published by these authors about these issues.

In spite of the large number of effects considered in Ref.[87] Matese and O'Connell did not include in their analysis a simpler and quantitatively more relevant effect of magnetic fields on the BBN, namely the direct contribution of the magnetic field energy density to the expansion rate of the Universe. The relevance of such effect was realized by Greenstein [89] shortly after the publication of the Matese and O'Connell paper. Greenstein showed that by increasing the Universe expansion rate the presence of the magnetic field also increases the temperature at which the neutron-proton equilibrium ratio is frozen. Since this ratio is roughly given by [46]

$$(n/p)_{eq} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp(-Q/T) \ , \qquad (3.1)$$

where $Q \equiv m_n - m_p$, a small change in the freezing temperature gives rise to a large variation in the neutron relative abundance hence in the relic abundance of the light elements. In his paper Greenstein also noted that if the magnetic field is sufficiently tangled over distances small compared to the events horizon, it will have no effect on the Universe geometry. An explicit calculations of the ^{4}He relic abundance as a function of the magnetic field strength were reported in a previous paper by the same author [90]. Greenstein concluded that the effect of the magnetic field energy density overcomes that of the magnetic field on the neutron decay discussed by Matese and O'Connell. Furthermore, from the requirement that the relic ^{4}He mass fraction does not exceed the 28%, he inferred the upper limit $B \lesssim 10^{12}$ Gauss at the time when $T = 5 \times 10^9$ ^{o}K.

In a following paper by Matese and O'Connell [88], the authors performed a more careful analysis of the effects of a magnetic field on the weak reactions

[1] This issue was previously considered by Thorne [54]

which keep neutron and protons in thermal equilibrium considering, this time, also the direct effect of the magnetic field on the Universe expansion rate. Their final conclusions were in agreement with Greenstein's result.

The recent activity about the origin of magnetic fields during phase transitions in the early Universe (see Chap.4) renewed the interest on the BBN bounds on primordial magnetic fields and induced several authors to reconsider the work of Matese and O'Connell and Greenstein. It is remarkable that after about twenty years and a large number of new astrophysical observations Greenstein's and Matese and O'Connell upper limit remains today roughly unchanged. Moreover, this is the case in spite of important developments of the BBN numerical computations codes.

We shall now abandon our historical approach to this section and proceed to give a more detailed description of the subject.

3.1 The effect of a magnetic field on the neutron-proton conversion rate

The reactions which are responsible for the chemical equilibrium of neutrons and protons in the early Universe are the weak processes

$$\begin{aligned} n \,+\, e^+ \quad &\leftrightarrow \quad p \,+\, \overline{\nu}_e && (3.2)\\ n \,+\, \nu_e \quad &\leftrightarrow \quad p \,+\, e^- && (3.3)\\ n \quad &\leftrightarrow \quad p \,+\, e^- \,+\, \overline{\nu}_e \,. && (3.4) \end{aligned}$$

In the absence of the magnetic field and in the presence of a heat-bath, the rate of each of the previous processes takes the generic form

$$\Gamma(12 \to 34) = \left(\prod_i \frac{\int d^3\mathbf{p}_i}{(2\pi)^3 2E_i}\right)(2\pi)^4\delta^4(\textstyle\sum_i p_i)|\mathcal{M}|^2 f_1 f_2(1-f_3)(1-f_4), \quad (3.5)$$

where p_i is the four momentum, E_i is the energy and f_i is the distribution function of the i-th particle species involved in the equilibrium processes. All processes (3.2,3.3,3.4) share the same amplitude $\mathcal{M}$ determined by the standard electroweak theory.

The total neutrons to protons conversion rate is

$$\Gamma_{n\to p}(B=0) = \frac{1}{\tau}\int_1^\infty d\epsilon \, \frac{\epsilon\sqrt{\epsilon^2-1}}{1+e^{\frac{m_e \epsilon}{T}}}\left[\frac{(q+\epsilon)^2 e^{\frac{(\epsilon+q)m_e}{T_\nu}}}{1+e^{\frac{(\epsilon+q)m_e}{T_\nu}}} + \frac{(\epsilon-q)^2 e^{\frac{\epsilon m_e}{T}}}{1+e^{\frac{(\epsilon-q)m_e}{T_\nu}}}\right] \quad (3.6)$$

where q and ϵ are respectively the neutron-proton mass difference and the electron, or positron, energy, both expressed in units of the electron mass m_e. We neglect here the electron and neutrino electron chemical potentials as these are supposed to be small quantities during BBN [46]. The rate $1/\tau$ is defined by

$$\frac{1}{\tau} \equiv \frac{G^2(1+3\alpha^2)m_e^5}{2\pi^3} \tag{3.7}$$

where G is the Fermi constant and $\alpha \equiv g_A/g_V \simeq -1.262$. For $T \to 0$ the integral in Eq.(3.6) reduces to

$$I = \int_1^q d\epsilon \epsilon (\epsilon - q)^2 \sqrt{\epsilon^2 - 1} \simeq 1.63 \tag{3.8}$$

and $\tau_n = \tau/I$ is the neutron time-life.

The total rate for the inverse processes ($p \to n$) can be obtained reversing the sign of q in Eq.(3.6). It is assumed here that the neutrino chemical potential is vanishing (at the end of Sec.3.4 it will also be discussed the case where such an assumption is relaxed). Since, at the BBN time temperature is much lower than the nucleon masses, neutrons and protons are assumed to be at rest.

As pointed out by Matese and O'Connell [86, 88], the main effect of a magnetic field stronger than the critical value B_c on the weak processes (3.2-3.4) comes-in through the effect of the field on the electron, and positron, wave function which becomes periodic in the plane orthogonal to the field [38]. As a consequence, the components of the electron momentum in that plane are discretized and the electron energy takes the form

$$E_n(B) = \left[p_z^2 + |e|B(2n+1+s) + {m_e}^2\right]^{\frac{1}{2}} \tag{3.9}$$

where we assumed $\mathbf{B}$ to be directed along the $\mathbf{z}$ axis. In the above, n denotes the Landau level, and $s = \pm 1$ if, respectively, the electron spin is along or opposed to the field direction. Besides the effect on the electron dispersion relation, the discretization of the electron momentum due to the magnetic field has also a crucial effect on the phase-space volume occupied by these particles. Indeed, in the presence of a field with strength larger than B_c the substitution

$$\int \frac{d^3p}{(2\pi)^3} f_{FD}(E_0) \quad \longrightarrow \quad |e|B \sum_{n=0}^{\infty} (2 - \delta_{n0}) \int \frac{dp_e}{(2\pi)^2} f_{FD}(E_n(B), T) \; , \tag{3.10}$$

has to be performed [91]. Since we only consider here magnetic fields which are much weaker than the proton critical value ($eB \ll m_p^2$), we can safely disregard any effect related to the periodicity of the proton wave function.

The squared matrix element for each of the reactions (3.2-3.4) is the same when the spin of the initial nucleon is averaged and the spins of the remaining particles are summed. Neglecting neutron polarization, which is very small for $B < 10^{17}$ G, we have [86]

$$\sum_{\text{spins}} |\mathcal{M}(n)| = \frac{\gamma}{\tau}\left[1 - \delta_{n0}\left(1 - \frac{p_z}{E_n}\right)\right] . \tag{3.11}$$

It is interesting to observe the singular behaviour when a new Landau level opens up ($E_n = p_z$). Such an effect is smoothed-out when temperature is increased [92].

Expressions (3.9) and (3.10) can be used to determined the rate of the processes (3.2-3.4) in a heat-bath and in the presence of an over-critical magnetic field . We start considering the neutron β-decay. One finds

$$\Gamma_{n\to pe\bar{\nu}}(\gamma) = \frac{\gamma}{\tau}\sum_{n=0}^{n_{max}}(2-\delta_{n0}) \times \int_{\sqrt{1+2(n+1)\gamma}}^{q} d\epsilon \frac{\epsilon}{\sqrt{\epsilon^2 - 1 - 2(n+1)\gamma}} \times \frac{e^{\frac{m_e \epsilon}{T}}}{1+e^{\frac{m_e \epsilon}{T}}} \frac{(q-\epsilon)^2 e^{\frac{m_e(q-\epsilon)}{T_\nu}}}{1+e^{\frac{m_e(q-\epsilon)}{T_\nu}}} \tag{3.12}$$

where $\gamma \equiv B/B_c$ and n_{max} is the maximum Landau level accessible to the final state electron determined by the requirement $p_z(n)^2 = q^2 - m_e^2 - 2neB > 0$. It is noticeable that for $\gamma > \frac{1}{2}(q^2-1)^2 = 2.7$ only the $n = 0$ term survives in the sum. As a consequence the β-decay rate increases linearly with γ above such a value. The computation leading to (3.12) can be readily generalized to determine the rate of the reactions (3.2) and (3.3) for $\gamma \neq 0$

$$\Gamma_{ne\to p\bar{\nu}}(\gamma) = \frac{\gamma}{\tau}\sum_{n=0}^{\infty}(2-\delta_{n0}) \times \int_{\sqrt{1+2(n+1)\gamma}}^{\infty} d\epsilon \frac{\epsilon}{\sqrt{\epsilon^2 - 1 - 2(n+1)\gamma}} \times \frac{1}{1+e^{\frac{m_e \epsilon}{T}}} \frac{(q+\epsilon)^2 e^{\frac{m_e(q+\epsilon)}{T_\nu}}}{1+e^{\frac{m_e(q+\epsilon)}{T_\nu}}} , \tag{3.13}$$

and

$$\Gamma_{n\nu\to pe}(\gamma) = \frac{\gamma}{\tau}\left[\sum_{n=0}^{\infty}(2-\delta_{n0}) \times \int_{\sqrt{1+2(n+1)\gamma}}^{\infty} d\epsilon \frac{\epsilon}{\sqrt{(\epsilon-\kappa)^2 - 1 - 2(n+1)\gamma}}\right.$$

$$\times \frac{e^{\frac{m_e \epsilon}{T}}}{1+e^{\frac{m_e \epsilon}{T}}} \frac{(\epsilon-q)^2 e^{\frac{m_e(q+\epsilon)}{T}}}{1+e^{\frac{m_e(\epsilon-q)}{T_\nu}}} \Bigg]$$

$$- \left[\sum_{n=0}^{n_{max}} (2-\delta_{n0}) \times \int_{\sqrt{1+2(n+1)\gamma}}^{q} d\epsilon \frac{\epsilon}{\sqrt{\epsilon^2-1-2(n+1)\gamma}} \right.$$

$$\left. \times \frac{e^{\frac{m_e \epsilon}{T}}}{1+e^{\frac{m_e \epsilon}{T}} \frac{(\epsilon-q)^2 e^{\frac{m_e(q-\epsilon)}{T}}}{1+e^{\frac{m_e(\epsilon-q)}{T_\nu}}}} \right] \quad (3.14)$$

By using the well know expression of the Euler-MacLaurin sum (see e.g. Ref.[91]) it is possible to show that in the limit $B \to 0$ Eqs.(3.12-3.14) reduces to the standard expressions derived in the absence of the magnetic field [2].

The global neutron to proton conversion rate is obtained by summing the last three equations

$$\Gamma_{n \to p}(\gamma) = \frac{\gamma}{\tau} \sum_{n=0}^{\infty} (2-\delta_{n0}) \times \int_{\sqrt{1+2(n+1)\gamma}}^{\infty} d\epsilon \frac{\epsilon}{\sqrt{(\epsilon-\kappa)^2-1-2(n+1)\gamma}}$$

$$\times \frac{1}{1+e^{\frac{m_e \epsilon}{T}}} \left[\frac{(\epsilon+q)^2 e^{\frac{m_e(\epsilon+q)}{T_\nu}}}{1+e^{\frac{m_e(\epsilon+q)}{T_\nu}}} + \frac{(\epsilon-q)^2 e^{\frac{m_e \epsilon}{T}}}{1+e^{\frac{m_e(\epsilon-q)}{T_\nu}}} \right] \quad (3.15)$$

It is noticeable that the contribution of Eq.(3.12) to the total rate (3.15) is canceled by the second term of (3.14). As a consequence it follows that Eq.(3.15) does not depend on n_{max} and the $n \to p$ conversion grows linearly with the field strength above B_c. From Fig.3.1 the reader can observe that, in the range considered for the field strength, the neutron depletion rate drops quickly to the free-field when the temperature grows above few MeV's. Such a behaviour is due to the suppression of the relative population of the lowest Landau level when $eB \gg T^2$.

In the absence of other effects, the consequence of the amplification of $\Gamma_{n \to p}$ due to the magnetic field would be to decrease the relic abundance of ^{4}He. In fact, a larger $\Gamma_{n \to p}$ implies a lower value of the temperature (T_F) at which the neutron to proton equilibrium ratio is frozen because of the expansion of the Universe. It is evident from (3.1) that the final value of (n/p) drops exponentially as T_F is increased. Furthermore, once n/p has

[2] For a different approach see Ref.[93]

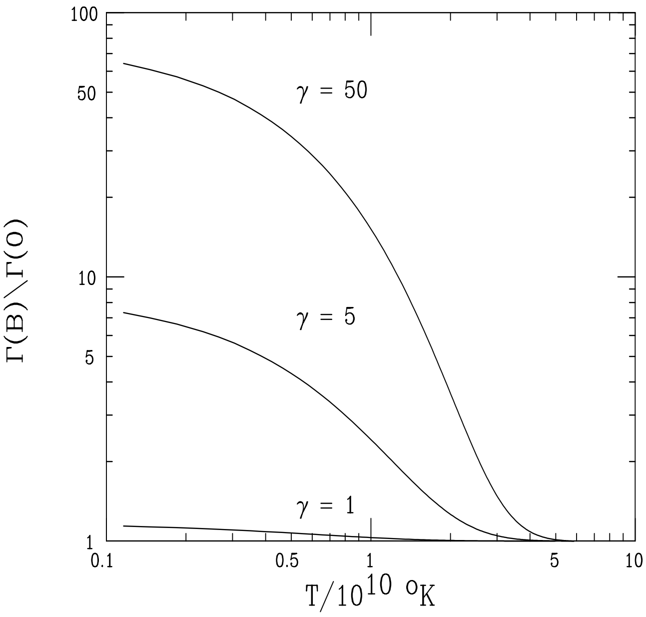

Figure 3.1: The neutron-depletion rate $\Gamma_{n\to p}$, normalized to the free-field rate, is plotted as a function of the temperature for several values of γ. From Ref.[94]

been frozen, occasional neutron β-decays can still reduce the relic neutron abundance [46]. As it follows from Eq.(3.12), the presence of a strong magnetic field accelerates the process which may give rise to a further suppression of the n/p ratio. In practice, however, neutron decay takes place at a times when the magnetic field strength has already decreased significantly due to the Universe expansion so that the effect is negligible.

The result of Matese and O'Connell has been confirmed by Cheng et *al.* [93] and by Grasso and Rubinstein [94]. Among other effects, the authors of Ref.[94] considered also QED and QCD corrections in the presence of strong magnetic fields . In principle these corrections may not be negligible in the presence of over-critical magnetic fields and their computation requires a proper treatment. In order to give to the reader a feeling of the relevance of this issue, we remind him the wrong result which was derived by O'Connell [39] by neglecting QED radiative correction to the electron Dirac equation in the presence of a strong magnetic field. By assuming the electron anomalous magnetic moment to be independent on the external field O'Connell found

$$E_n = \left[p_z^2 + |e|B(2n+1+s) + {m_e}^2\right]^{\frac{1}{2}} + s\frac{\alpha}{2\pi}m_e\gamma \ . \tag{3.16}$$

For $B > (4\pi/\alpha)B_c$ this expression give rise to negative values of the ground state energy which, according to O'Connell, is the manifestation of the instability of the vacuum to spontaneous production of electron-positron pairs. This conclusion, however, is in contradiction with standard electrodynamics from which we know that a constant magnetic field cannot transfer energy. This problem was solved by several authors (see e.g. Ref.[37]) by showing that by properly accounting for QED radiative corrections to the Dirac equation no negative value of the electron energy appear. The effect can be parametrized by a field dependent correction to the electron mass, $m_e \to m_e + M$, where

$$M = \begin{cases} -\dfrac{\alpha}{2\pi}\dfrac{eB}{2m_e}\left[1 - \dfrac{8}{3}\dfrac{eB}{m_e^2}\left(\log\dfrac{m_e^2}{2eB} - \dfrac{13}{24}\right)\right] & B \ll B_c \\ \dfrac{\alpha}{4\pi}m_e\left(\log\dfrac{2eB}{m_e^2}\right)^2 \ . & B \gg B_c \end{cases} \tag{3.17}$$

Such a correction was included in Refs.[94, 95]. It is interesting to observe that although pair production cannot occur at the expense of the magnetic field , this phenomenon can take place in a situation of thermodynamic equilibrium where pair production can be viewed as a chemical reaction $e^+ + e^- \leftrightarrow \gamma$, the magnetic field playing the role of a catalysts agent [96]. We will return on this issue in Sec.3.3.

Even more interesting are the corrections due to QCD. In fact, Bander and Rubinstein showed that in the presence of very strong magnetic fields the neutron-proton effective mass difference q becomes [97] (for a more detailed discussion of this issue see Chap.5)

$$Q(B) = 0.12\mu_N B - m_n + m_p + f(B) \ . \tag{3.18}$$

The function $f(B)$ gives the rate of mass change due to colour forces being affected by the field. μ_N is the nucleon magnetic moment. For nucleons the main change is produced by the chiral condensate growth, which because of the different quark content of protons and neutrons makes the proton mass to grow faster[97]. Although, as a matter of principle, the correction to Q should be accounted in the computation of the rates that we reported above, in practice however, the effect on the final result is always negligible. More subtle it is the effect of the correction to Q on the neutron-to-proton equilibrium ratio. In fact, as it is evident from Eq.(3.1), in this case the correction to Q enters exponentially to determine the final neutron-to-proton ratio. However, the actual computation performed by Grasso and Rubinstein [94] showed that the effect on the light element abundances is sub-dominant whenever the field strength is smaller than $\lesssim 10^{18}$ Gauss.

3.2 The effects on the expansion and cooling rates of the Universe

In the previous section we discussed how the presence of strong magnetic fields affects the rates of the weak reactions which are responsible for the chemical equilibrium of neutrons and protons before BBN. The knowledge of such rates is, however, not sufficient to predict the relic abundances of the elements synthesized during BBN. In fact, the temperature T_F at which $(n/p)_{eq}$ is frozen is determined by the competition of the weak reaction and the Universe expansion according to the condition [46]

$$\Gamma_{n \to p}(T_F) = H(T_F) \ . \tag{3.19}$$

From this expression it is clear that in order to determine T_F the knowledge of the Universe expansion rate $H(T)$ is also required.

In the absence of a cosmological term and assuming the effect of the magnetic field on the Universe geometry to be negligible, H is determined by

$$H \equiv \frac{\dot{a}}{a} = \left(\frac{8\pi G \rho_{tot}}{3}\right)^{1/2} \ . \tag{3.20}$$

where, according to the standard notation, a is the scale factor of the Universe, G the Newton gravitational constant and $\rho(T)$ is the energy density of the Universe. In the presence of a magnetic field

$$\rho(T, B) = \rho_{em}(T, B) + \rho_\nu + \rho_B(B) \tag{3.21}$$

where $\rho_{em}(T, B)$ is the energy density of the standard electromagnetic component (photons + electrons and positrons) of the heat-bath and ρ_ν is the energy density of all neutrino species (the reason why ρ_{em} depends on the magnetic field strength will be discussed in the next section). We see that to the standard components of ρ it adds now the contributions of the magnetic field energy density

$$\rho_B(T) = \frac{B^2(T)}{8\pi} \, . \tag{3.22}$$

It is worthwhile to observe that, concerning its direct contribution to ρ, the magnetic field behaves like any relativistic component of the heat-bath. In fact, by assuming that the field is not too tangled on scales smaller than the magnetic dissipation scale, and that the Universe geometry is not affected by the magnetic field (see Sec.2.1), the magnetic flux conservation during the Universe expansion implies

$$B \propto R^{-2} \propto T^2 \quad \longrightarrow \quad \rho_B(T) \propto T^4 \ , \tag{3.23}$$

which is the same behaviour of the radiation.

In the absence of other effects, the relation (3.23) would allow to parametrize the effect of the magnetic field in terms of a correction to the effective number of massless neutrino species ΔN_ν^B [98]. Indeed, by comparing the contribution of N_ν light ($m_\nu \ll 1$ MeV) neutrino species with the energy density of the Universe, which is

$$\rho_\nu = \frac{7\pi^2}{120} N_\nu T_\nu^4 \ , \tag{3.24}$$

with (3.22) one gets

$$\Delta N_\nu^B = \frac{15}{7\pi^3} b^2, \tag{3.25}$$

where $b \equiv B/T_\nu^2$.

Before closing this section we have to mention another possible consequence of the faster Universe expansion induced by the presence of the magnetic field. The effect is due to shortening of time between weak reactions freeze-out and breaking of the deuterium bottleneck. It follows that neutrons have less time to decay before their confinement into nucleons take place which turns into a larger abundance of ^{4}He. In Ref.[98] it was showed that such an effect is generally sub-dominant.

3.3 The effect on the electron thermodynamics

In the above we discussed how the phase-space of electrons and positrons is modified by the presence of strong magnetic fields and how this effect changes the weak processes rates. The consequences of the variation of the electron phase-space, however, go well beyond that effect. Electron and positron thermodynamics functions will also be affected. In fact, by applying the prescription (3.10), we find that the number density, the energy density and the pressure of the electron-positron gas are now given by

$$
\begin{aligned}
n_e(B) &= \frac{eB}{(2\pi)^2}\sum_{n=0}^{\infty}(2-\delta_{n0})\int_{-\infty}^{+\infty} f_{FD}(T)dp_z && (3.26)\\
\rho_e(B) &= \frac{eB}{(2\pi)^2}\sum_{n=0}^{\infty}(2-\delta_{n0})\int_{-\infty}^{+\infty} E_n f_{FD}(T)dp_z && (3.27)\\
p_e(B) &= \frac{eB}{(2\pi)^2}\sum_{n=0}^{+\infty}(2-\delta_{n0})\int_{-\infty}^{+\infty} \frac{E_n^2-m_e^2}{3E_n} f_{FD}(T)dp_z && (3.28)
\end{aligned}
$$

where

$$f_{FD}(T) \equiv \frac{1}{1+e^{\beta E_n(p_z)}} \tag{3.29}$$

is the Fermi-Dirac distribution function, and $E_n(p_z)$ is given by (3.9). As for the case of the weak processes rates, it is possible to show that Eqs.(3.26-3.28) reduce to the their well know standard from in the limit $B \to 0$ (see e.g. Ref.[98]).

Numerical computations [94] show that, for small T, ρ_e grows roughly linearly with B when $B > B_c$. This effect is mainly due to - 1) the reduction, for each Landau level, of the area occupied by the cyclotron motion of the electron in plane perpendicular to the field; 2) the growth of the energy gap among the lowest Landau level and the $n > 0$ levels, which produces an overpopulation of the lowest Landau level. The first effect is the dominant one. The net number density and the pressure of the electron-positron gas follow a similar behaviour. As we already noted in Sec.3.1, the energy cost of producing the electron-positron pairs excess cannot be paid by the magnetic field which is supposed to be static. Rather, the "power bill" is paid from the heat-bath, or better, from its photon component [96, 99]. Especially in the context of BBN this point is quite relevant since the energy transfer from the photons to the lowest Landau level of the electron-positron gas will affect the expansion rate of the Universe, its cooling rate and the effective baryon-to-photon ratio η [94, 95, 98]. We start discussing the first two effects. We observe that the growth of the electron and positron energy density, due to the presence of the magnetic field, gives rise to a faster expansion rate of the Universe. This point was first qualitatively discussed by Matese and O'Connell [87] and recently analyzed in more detail by Grasso and Rubinstein [94]. The time-temperature relation will also be modified. The relevance of latter effect has been first showed by Kernan, Starkman and Vachaspati [98] by solving numerically the relation

$$\frac{dT}{dt} = -3H\frac{\rho_{em} + p_{em}}{d\rho_{em}/dT} \ , \tag{3.30}$$

where $\rho_{em} \equiv \rho_e + \rho_\gamma$ and $p_{em} \equiv p_\gamma + p_e$ are the energy density and the pressure of the electromagnetic component of the primordial heat-bath. In agreement with our previous considerations, Eq.(3.30) has been obtained by imposing energy conservation of the electromagnetic component plasma.

For small values of the ratio eB/T^2, the most relevant effect of the magnetic field enters in the derivative $d\rho_{em}/dT_\gamma$ that is smaller than the free field value. This effect can be interpreted as a delay in the electron-positron annihilation time induced by the magnetic field. This will give rise to a slower entropy transfer from the electron-positron pairs to the photons, then to a slower reheating of the heat bath. In fact, due to the enlarged phase-space of the lowest Landau level of electrons and positrons, the equilibrium of the process $e^+e^- \leftrightarrow \gamma$ is shifted towards its left side. Below we will discuss as this effect has a clear signature on the deuterium and ^{3}He predicted abundances. Another point of interest is that the delay in the e^+e^- annihilation causes a slight decrease in the T_ν/T ratio with respect to the canonical value, $(4/11)^{1/3}$ [98].

The delay in the entropy transfer from the e^+e^- gas to the heat-bath induces also an increment in the value of baryon-to-photon ratio η. In the absence of other effects a larger value of η would induce smaller relic abundances of Deuterium and ^{3}He. This effect, was first predicted in the Refs.[94, 95], Furthermore, it is interesting to observe that in the case the primordial magnetic field is inhomogeneous and it is confined in finite volume regions where its strength exceed the cosmic mean value (e.g. flux tubes), this effect may give rise to spatial variation in the relic element abundances.

3.4 Derivation of the constraints

In order to account for all the effects discussed in the previous sections, the use of a numerical analysis is required. Usually, this is done by modifying properly the famous BBN numerical code developed by Wagoner and improved by Kawano [100]. After some discussion around the relative importance of the different effects, the results of different groups have converged to a common conclusion: the most relevant effect of a cosmological magnetic field on BBN is that produced by the energy density of the field on the Universe expansion rate. This is in qualitative agreement with the early result of Greenstein [89]. From a more quantitative point of view, however, the effect of the magnetic field on the electron thermodynamics cannot be totally neglected. In fact, it was showed in [95] that such an effect produces sizable changes in the relic abundance of ^{4}He, Deuterium and ^{3}He (see e.g. Fig.3.2 for the ^{4}He relic abundance prediction) As a consequence, we think that the effect of the magnetic field on the BBN cannot be simply parameterized in terms of a contribution to the effective number of neutrino species. Although, in this respect, a different conclusion was reached in [98, 101] it should be noted that, differently from [95], in those papers only approximate expressions for the electron thermodynamic quantities in the presence of a strong magnetic field were used. Such an approximation may be not justified when $eB \gtrsim T^2$.

According to the standard procedure, the upper limit on the strength of the cosmological magnetic field was obtained in [94, 95, 102] by comparing the numerically predicted relic abundance of ^{4}He with the observational upper limit. In [95] however, the information about Deuterium and ^{3}He was also used. In fact, since the effective value of η is also affected by the magnetic field , η was chosen in the actual numerical simulation so to saturate the predicted value of D $+^3$ He/H to the observational upper limit. This choice assured the minimal predicted abundance of ^{4}He for each considered value of B.

By fixing $N_\nu = 3$, requiring $Y_P < 0.245$ and D $+^3$ He/H $< 1.04 \times 10^{-4}$ Grasso and Rubinstein derived the upper limit

$$B(T = 10^9 \;\; \mathrm{K}) \lesssim 1 \times 10^{13}\ \mathrm{G}\ . \tag{3.31}$$

Similar results has been obtained by the authors of Refs.[101, 103].

It is useful to remind to the reader under which assumptions the previous limit has been derived. They are the following.

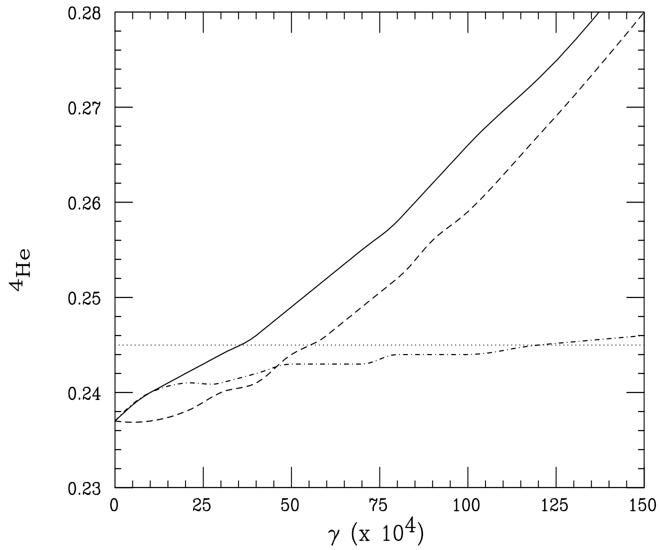

Figure 3.2: The ^{4}He predicted abundance is represented in function of the parameter γ, considered at $T = 10^9\ ^oK$, in three different cases: only the effect of the magnetic field energy density is considered (dashed line); only the effect of the field on the electron statistics is considered (dotted-dashed line); both effects are considered (continuous line). The dotted line represents the observational upper limit. From Ref.[95]

1. Universe dynamics was assumed to be well described by Friedman-Robertson-Walker metric. In other words we are assuming that the magnetic field does not lead to a sizeable anisotropic component in the Universe expansion [3].

2. The effective number of neutrino species is three. This means, for example, that if neutrinos are of Dirac type their mass and magnetic moment are negligible. This in order do not populate right handed degree of freedom by the magnetic dipole interaction of the neutrinos with the field.

3. The neutrino chemical potential is negligible.

4. Fundamental physical constants are equal to their present time values (for a discussion on this issue see Ref.[104]).

[3] BBN in the presence of anisotropic Universe, possibly due to a homogeneous cosmic magnetic field, has been considered by Thorne [54]

Some of these assumptions will be relaxed in the following part of this chapter.

In order to translate our limit (3.31) into a bound on the magnetic field at the time of galaxy formation some caution is required. If we just assume that the magnetic field re-scales adiabatically with the Universe expansion, according to Eq.(3.23), the BBN limit reads

$$B_0 \lesssim 7 \times 10^{-5}\ \mathrm{G}\ . \tag{3.32}$$

We should keep in mind, however, that in this case we are neglecting any possible nonadiabatic evolution of the magnetic field as that which could be induced by a non-trivial topology of the field. Even assuming an adiabatic evolution, we note that the limit (3.32) cannot be directly interpreted as a limit on the progenitor of galactic magnetic fields. The reason for that is that BBN probes magnetic fields on scales of the order of the horizon radius at BBN time (the Hubble comoving radius at BBN time is ~ 100 pc) which are much smaller than typical protogalaxy sizes ($\sim 1 - 10$ Mpc). Therefore, if cosmic magnetic field are tangled on scales smaller than the protogalactic size, the progenitor magnetic field has to be interpreted as a proper average of smaller flux elements. Clearly, the result of such an average will depend on the statistical properties of the random magnetic field. If the field vector were to perform a random walk in 3d volume, the scaling would be $B(L) \equiv \langle B \rangle_{\mathrm{rms},\ L} \sim N^{-3/2}$ [105], where L_0 is the comoving coherence length of the magnetic field and $N = L/L_0$ is the number of steps. An argument based on the statistical independence of conserved flux elements gives $B(L) \sim N^{-1}$ [106], whereas another argument based on the statistical independence of the field in neighboring cells predicts $B(L) \sim N^{-1/2}$ [107]. Adopting a phenomenological point of view, one may just write that the rms field computed on the scale L at the time t is [108]

$$\langle B(L,t) \rangle_{\mathrm{rms}} = B_0 \left(\frac{a_0}{a(t)} \right)^2 \left(\frac{L_0}{L} \right)^p\ , \tag{3.33}$$

where p is an unknown parameter ($p = 3/2, 1, 1/2$ respectively in the three cases discussed in the above). The meaning of B_0 is now understood as $B_0 = \lim_{L \to \infty} B(L, t_0)$ [4]. If, for example, we adopt the value $p = 1$ and

[4] A detailed discussion about average procedures of tangled magnetic fields can be found in Ref.[109]

assume $L_0 = 100$ pc, the limit (3.32) implies

$$\langle B(1\ \mathrm{Mpc}, t_0)\rangle_{\mathrm{rms}} \lesssim 10^{-9}\ \mathrm{G}\ . \tag{3.34}$$

Therefore, although the BBN bound is much more stringent than what is usually claimed in the literature, it cannot exclude a primordial origin of galactic magnetic fields by the adiabatic compression of the field lines.

For the same reasons which we have explained in the above, BBN limits on primordial magnetic fields cannot be directly compared with bounds derived from the analysis of CMBR anisotropies. In fact, unless the magnetic fields is uniform through the entire Universe, CMBR offers a probe of magnetic fields only on comoving scales which are much larger than the horizon radius at BBN time.

We shall now consider how the previous limits changes by relaxing one of the assumptions under which the constraint (3.31) has been derived, namely that related to the neutrino chemical potential. The effects of a possible neutrino-antineutrino asymmetry in this context has been recently considered by Suh and Mathews [110]. This issue is interesting since several recent leptogenesis scenarios predict the formation of such asymmetry during the radiation era. It is well know that even in the absence of a primordial magnetic field a non-vanishing neutrino chemical potential can affect the predictions of BBN (see Ref.[111] and references therein). In fact, a degeneracy of the electron neutrino changes both the weak reaction rates and the neutron-to-proton equilibrium ratio, whereas a degeneracy in any of the neutrino species modifies the expansion rate of the Universe. Clearly, the presence of any of these effects would affect the BBN limit on the strength of a primordial magnetic field . Suh and Mathews found that if the limit is $B_0 \leq 5.8 \times 10^{-7}$ G with $\xi_e \equiv \mu_{\nu_e}/T_{\nu_e} = 0$ (in good agreement with the limit (3.32)), it becomes $B_0 \leq 2.8 \times 10^{-6}$ Gauss if $\xi_e \equiv \mu_{\nu_e}/T_{\nu_e} = 0.15$. Therefore, we see that in the presence of phenomenologically acceptable values of the neutrino chemical potential the BBN constraint on the magnetic field can be considerably relaxed.

3.5 Neutrino spin-oscillations in the presence of a magnetic field

It is interesting to consider how the limit obtained in the previous section changes if neutrinos carry non-standard properties which may change the effective neutrino number during BBN. We are especially interested here to the possibility that neutrinos carry non-vanishing masses and magnetic moments. If this is the case, the dipole interaction of the neutrinos with the magnetic field may give rise to *spin-oscillations* of the neutrinos, i.e. periodic conversion of a helicity state into another. In the case of Dirac neutrinos, this phenomenon may have crucial consequences for BBN. In fact, spin-oscillation can populate the right-handed helicity state of the neutrino which, being practically sterile (for $m_\nu \ll T$) to weak interactions, would otherwise play no effective role. By adding a new degree of freedom to the thermal bath, such an effect may produce dangerous consequences for the outcome of BBN. This problem was first pointed-out by Shapiro and Wasserman [112] and, independently, by Lynn [113] who used the argument to put a constraint on the product of the magnetic field with the neutrino magnetic moment. In both works, however, the important role played by neutrino refractive properties determined by the neutrino collective interaction with the heat-bath, as well as that played by neutrino scattering with leptons, were disregarded. A more complete treatment was developed by Fukugita et al. [114]. They showed that the conditions under which the neutrino wrong-helicity state can be effectively populated are the following:

1. the spin-oscillation frequency $\Delta E_{\rm magn} = 2\mu_\nu B$ has to exceed the Universe expansion rate;

2. since neutrino scattering destroy the phase relationship between the left-handed and right-handed helicity states, $\Delta E_{\rm magn}$ has to be larger than the scattering rate;

3. since the refractive indices for left-handed and right-handed states, n_L and n_R, are not equal, oscillations can occur only if

$$\Delta E_{\rm magn} \lesssim \Delta E_{\rm refr} \tag{3.35}$$

where $\Delta E_{\rm refr} \equiv (n_L - n_R)E_\nu$.

The BBN is affected only if such conditions are simultaneously satisfied at some temperature $T_{\rm osc}$ in the range $T_{\rm dec} \lesssim T_{\rm osc} \lesssim T_{\rm QCD}$ where $T_{\rm dec} \approx 1$ MeV is the neutrino decoupling temperature and $T_{\rm QCD} \approx 200$ MeV. Note that in the case right-handed neutrinos decouples before the QCD phase transition, the huge amount of entropy which is expected to be released during this transition would dilute their relative abundance so to prevent any effect on the BBN. From the previous considerations the authors of Ref.[114] derived the limit

$$\mu_\nu \lesssim 10^{-16}\ \mu_B \left(\frac{10^{-9}\ {\rm G}}{B_0}\right) , \tag{3.36}$$

where μ_B is the Bohr magneton. The work of Fukugita et al. has been reconsidered by several authors. For example, Enqvist, Olesen and Semikoz [115], improved the previous analysis by considering the effect of the neutrino refractive properties on the left-right transition probability. Elmfors, Grasso and Raffelt [116] accounted for the effect of the magnetic field on the neutrino refractive properties and used an improved treatment of neutrino collisions. First of all, Elmfors et al. noted that by affecting the thermodynamics properties of the electromagnetic component of the heat-bath (see Sec.3.3) a strong magnetic field changes also the neutrino potentials. This may have relevant consequences both for neutrino spin-oscillations and flavour oscillations in a magnetized medium [117]. The interplay between spin-oscillations and collisions was then accounted in Ref.[116] by means of the following evolution equation [118]

$$\frac{\partial \mathbf{P}}{\partial t} = \mathbf{V} \times \mathbf{P} - D\mathbf{P}_{\rm T} \ , \tag{3.37}$$

where $\mathbf{P}$ is the neutrino polarization vector and $\mathbf{P}_{\rm T}$ is its transverse component respect to the neutrino direction of motion. $\mathbf{V}$ is a vector of effective magnetic interaction energies which can be decomposed into its transverse and longitudinal components

$$\mathbf{V}_{\rm T} = 2\mu_\nu \mathbf{B}_{\rm T} , \tag{3.38}$$

$$|\mathbf{V}_{\rm L}| = \Delta E_{\rm refr} , \tag{3.39}$$

where μ_ν is the neutrino magnetic moment and [116]

$$\Delta E_{\rm refr}(B) = \frac{8\sqrt{2}\, G_{\rm F} E}{3} \sum_{\ell=e,\mu,\tau} \left(\frac{\rho^L_{\nu_\ell}(B)}{m_Z^2} + \frac{\rho_\ell}{m_W^2}\right) \tag{3.40}$$

is the left-right neutrino energy difference in the magnetized medium [5]. It is worthwhile to note that in Eq.(3.40) the expression (3.27) has to be used for $\rho_e(B)$. As we wrote above, collisions destroy the phase coherence between the left-handed and right-handed component of a neutrino state, which amounts to a damping of the transverse part $\mathbf{P}_{\rm T}$ of the polarization vector. The main contribution to the damping rate D comes from neutrino elastic and inelastic scattering with leptons and equals half the total collision rate of the left-handed component [121]. In the early Universe at $T \sim 1$ MeV one finds:

$$\langle D \rangle = f_{\rm D} \frac{7\pi}{16} G_{\rm F}^2 T^5 \ , \tag{3.41}$$

where $f_{\rm D}$ is a order one numerical factor. Inserting the previous expressions in Eq.(3.37) it is easy to derive the neutrino depolarization rate $\Gamma_{\rm depol}$. In the small mixing angle limit,

$$\tan 2\theta = \frac{V_{\rm T}}{V_{\rm L}} = \frac{2\mu B_{\rm T}}{\Delta E_{\rm refr}} \ll 1 \ , \tag{3.42}$$

one finds

$$\Gamma_{\rm depol} \approx \frac{(2\mu B_{\rm T})^2 \, \langle D \rangle}{\langle V_L^2 \rangle} \approx \frac{f_{\rm D}}{f_{\rm L}^2} \, \frac{400\, \alpha^2}{7\pi} \, \frac{\mu_\nu^2 B_{\rm T}^2}{G_{\rm F}^2 T^5} \ , \tag{3.43}$$

where $f_L = 1$ for μ and τ neutrinos, while for e neutrinos $f_L \approx 3.6$. By requiring this rate to be smaller than the Universe expansion rate $H(T)$ in the temperature interval $T_{\rm dec} < T < T_{\rm QCD}$, Elmfors at al. [116] found the upper limit [6]

$$\mu_\nu \lesssim 7 \times 10^{-17} \ \mu_B \ \left(\frac{10^{-9} \ {\rm G}}{B_0} \right) \ , \tag{3.44}$$

which is not too different from the limit (3.36) previously found by the authors of Ref.[114]. Limits on μ_ν were also found by the authors of Refs.[108, 122] who considered the case of random magnetic fields.

In principle, right-handed neutrinos could also be populated by direct spin-flip interactions mediated by virtual photons produced by scattering on charged particles or by annihilation processes [123], as well as by the interaction with small scale magnetic fields produced by thermal fluctuations [124]. In practice, however, bounds on μ_ν from a possible large scale magnetic

[5] For a computation of the neutrino refractive properties in a magnetized medium see also Refs.[119, 120].

[6] Note that in Ref. [116] B_0 was defined as the magnetic field at BBN time.

field are found to be more stringent even for very weak magnetic fields. The most stringent upper limit on Dirac type neutrino magnetic moment with mass $m_\nu < 1$ MeV, comes from stellar evolution considerations. It is $\mu_\nu \lesssim 3 \times 10^{-12} \mu_{\rm B}$ [125, 126]. It is interesting to observe that if one of the neutrinos saturate this limit, Eq.(3.44) implies the following quite stringent bound on the present time cosmic magnetic field, $B_0 \lesssim 10^{-13}$ G.

In the particle physics standard model, neutrinos have no magnetic dipole moment. However, if the neutrino has a Dirac mass m_ν, radiative corrections automatically give rise to a finite dipole moment [113]

$$\mu_\nu = 3.2 \times 10^{-19} \ \mu_{\rm B} \ \left(\frac{m_\nu}{1 {\rm eV}} \right) \ , \tag{3.45}$$

even without invoking any further extension of the standard model beside that required to account for the finite neutrino mass. On the basis of this consideration Enqvist et al. [127] derived the following upper limit for the present time local magnetic field

$$B_0 \lesssim \frac{2 \times 10^{-3} \ {\rm G}}{\displaystyle\sum_i \frac{m_{\nu_i}}{1 \ {\rm e}V}} \ . \tag{3.46}$$

Clearly, this limit cannot compete with the constraint derived in the previous section.

Spin oscillations in the presence of twisted primordial magnetic fields (i.e. magnetic field with a nonvanishing helicity, see Sec.1.4) have been considered by Athar [128]. Athar showed that in such a situation the left-right conversion probabilities for neutrino and antineutrinos may be different. This result may open the interesting possibility that a neutrino-antineutrino asymmetry may be generated during the big-bang by a preexisting non trivial topology of a primeval magnetic field. As we shall see in Sec.4.4, the production of a net magnetic helicity of the Universe is indeed predicted by some models.

It is also interesting to speculate on the effects when the number of dimensions change, and these are large [129]. In fact BBN is one of the most serious objections to this idea, together with the background diffuse gamma radiation [130]. Detailed studies of effects of magnetic fields in these scenarios are not available yet.

Chapter 4

Generation of magnetic fields

4.1 Magnetic fields from primordial vorticity

The idea that primordial magnetic fields can be produced by plasma vortical motion during the radiation era of the early Universe has been first proposed by Harrison [131]. Since this mechanism has been reviewed in several papers (see e.g. [1, 132]) we shall not discuss it in detail here. Harrison's mechanism is based on the consideration that the electron and ion rotational velocities should decrease differently in the expanding Universe in the pre-recombination era. The reason is that Thomson scattering is much more effective for electrons than ions. Therefore electrons remains tightly coupled for a longer time to the radiation and behave like relativistic matter whereas ions are already non-relativistic. It follows that during Universe expansion angular velocity decreases like $\omega \propto a^{-1}$ for electrons and like $\omega \propto a^{-2}$ for ions, where a is the Universe scale factor. The difference between these two velocities causes an electromotive force, hence an electric current which generate a magnetic field. Harrison [133] showed that if a primordial turbulence was present at the recombination this mechanism may lead to present time intergalactic magnetic fields as large as 10^{-8} G on a scale-length of 1 Mpc.

A problem, however, arises with this scenario. In fact, it was noted by Rees [19] that since rotational, or vector, density perturbations decay with cosmic expansion, in order to produce sizeable effects at recombination time this kind of fluctuations should has been dominant at the radiation-matter equality time. This seems to be incompatible with the standard scenario for galaxy formation.

Another related problem is that, in contrast to scalar or tensor perturbations, rotational perturbations cannot arise from small deviations from the isotropic Friedmann Universe near the initial singularity. This is a consequence of the Helmholtz-Kelvin circulation theorem which states that the circulation around a closed curve following the motion of matter is conserved. Such a problem, however, may be partially circumvented if collisionless matter was present during the big-bang (e.g. decoupled gravitons after the Planck era). Rebhan [68] showed that in this case the Helmholtz-Kelvin theorem does not apply and growing modes of vorticity on superhorizon scale can be obtained. In fact, a non perfect fluid can support anisotropic pressure which may generate nonzero vorticity even if it was zero at the singularity. It follows that the only constraint to the amount of primordial vorticity comes from the requirement that it does not produce too large anisotropies in the CMBR. Rebhan showed that this requirement implies the following upper limit to the strength of a present time intergalactic magnetic field, with coherence length L, produced by vortical plasma motion

$$B_0(L) < 3 \times 10^{-18}\ h^{-2}\ L_{\rm Mpc}^{-3}\ {\rm G}\ . \tag{4.1}$$

Such a field might act as a seed for galactic dynamo.

If primordial vorticity is really not incompatible with standard cosmology, another interesting possibility is to generate primordial magnetic fields arises. It was noted by Vilenkin [134] that, as a consequence of parity violation in the Weinberg-Salam model of the electroweak interactions, macroscopic parity-violating currents may develop in a vortical thermal background. Vilenkin and Leavy [135] suggested that this currents may effectively give rise to strong magnetic fields. It was also recently noted by Brizard, Murayama and Wurtele [136] that in the presence of vorticity and a neutrino-antineutrino asymmetry, collective neutrino-plasma interactions may power astrophysical as well as cosmological magnetic fields.

Clearly, in order to implement these scenarios, a suitable mechanism to produce the required amount of primordial vorticity has to be found. Among other exotic possibilities, generation of vorticity and magnetic fields by the anisotropic collapse of conventional matter into the potential well of pre-existing dark matter condensations in a texture-seeded scenario of structure formation [132], and from rotating primordial black-holes [135] have been considered in the literature.

In our opinion, primordial phase transitions, may provide a more realistic source of vorticity. The generation of primeval magnetic fields during some of these transitions will be the subject of the next sections of this chapter.

4.2 Magnetic fields from the quark-hadron phase transition

It is a prediction of Quantum-Chromo-Dynamics (QCD) that at some very high temperature and/or density strongly interacting matter undergoes a deconfinement transition, where quark/gluon degrees of freedom are "melted-out" from hadrons. In the early Universe, due to the cosmological expansion, the process proceeds in the opposite direction starting from a "quark-gluon plasma" which at some critical temperature T_{QCD} condenses into colorless hadrons [137]. Lattice computations suggest that the QCD phase transition (QCDPT) is a first-order phase transition taking place at $T_{QCD} \sim 150$MeV [138]. Typically a first-order phase transition takes place by bubble nucleation. As the temperature supercools below T_{QCD}, sub-critical bubbles containing the hadronic phase grow as burning deflagration fronts releasing heat in the quark-gluon plasma in the form of supersonic shock fronts. When the shock fronts collide they reheat the plasma up to T_{QCD} stopping bubble grow. Clearly, up to this time, the transition is an out-of-equilibrium process. Later on, the growth of newly nucleated bubbles proceeds in thermal equilibrium giving rise to the, so-called, coexistence phase. The latent heat released by these bubbles compensate for the cooling due to the Universe expansion keeping temperature at T_{QCD}. The transition ends when expansion wins over and the remaining quark-gluon plasma pockets are hadronized.

The first step of the magnetogenesis scenario at the QCDPT proposed by Quashnock, Loeb and Spergel [139] consists in the formation of an electric field behind the shock fronts which precede the expanding bubbles. This is a consequence of the baryon asymmetry, which was presumably already present and which makes the baryonic components of the primordial plasma positively charged. At the same time, the leptonic component must be negatively charged to guarantee the Universe charge neutrality. The other crucial ingredient of the mechanism is the difference in the equation of state of the baryonic and leptonic fluids. As a consequence, the strong pressure gradient produced by the passage of the shock wave gives rise to a radial electric field behind the shock front. Such a generation mechanism is usually known as a battery. Quashnock et al. gave the following estimate for the strength of the electric field

$$eE \simeq 15 \left(\frac{\epsilon}{10\%}\right) \left(\frac{\delta}{10\%}\right) \left(\frac{kT_{QCD}}{150 \text{ MeV}}\right) \left(\frac{100 \text{ cm}}{\ell}\right) \frac{\text{keV}}{\text{cm}} , \tag{4.2}$$

where ϵ represents the ratio of the energy densities of the two fluids, $\delta \equiv (\ell \Delta p/p)$ is the pressure gradient and ℓ is the average distance between nucleation sites.

Small scale magnetic fields are generated by the electric currents induced by the electric fields. These fields, however, live on a very small scale ($\ll \ell$)and presumably they are rapidly dissipated. Phenomenologically more interesting fields should be produced when the shock fronts collide giving rise to turbulence and vorticity on a scales of order ℓ. Magnetic fields are produced on the same scale by the circulation of electric fields of magnitude given by Eq.(4.2). Then, using standard electrodynamics, Quashnock et al. found that the magnetic field produced on the scale $\ell \sim 100$ cm has a magnitude

$$B_\ell \simeq vE \simeq 5 \text{ G} . \tag{4.3}$$

Following the approach which was first developed by Hogan [105], the magnetic field on scales $L \gg \ell$ can be estimated by performing a proper *volume average* of the fields produced by a large number of magnetic dipoles of size ℓ randomly oriented in space. Such an average gives

$$B_L = B_\ell \left(\frac{\ell}{L}\right)^{3/2} . \tag{4.4}$$

After the QCDPT the magnetic field evolves according to the frozen-in law (1.24). It is straightforward to estimate the magnetic field strength at the recombination time on a given scale L. The smaller conceivable coherence length is given by the dissipation length at that time which, following the argument already described in Sec.1.4, is found to be $L_{diss}(t_{rec}) \simeq 5 \times 10^{10}$ cm, corresponding to 1 A.U. at present time. On this scale the magnetic field produced at the QCDPT is $\simeq 2 \times 10^{-17}$ G. This small strength is further dramatically suppressed if one considers scales of the order of the galactic size ~ 10 kpc. Therefore, it looks quite unplausible that the magnetic fields generated by this mechanism could have any phenomenological relevance even if the galactic dynamo was effective.

According to a more recent paper by Cheng and Olinto [140] stronger fields might be produced during the coexistence phase of the QCDPT. The new point raised by the authors of Ref.[140] is that even during such equilibrium phase a baryon excess build-up in front of the bubble wall, just as a consequence of the difference of the baryon masses in the quark and hadron phases. According to some numerical simulations [141], this effect might enhance the baryon density contrast by few orders of magnitude. Even more relevant is the thickness of the charged baryonic layer which, being controlled by baryon diffusion, is $\sim 10^7$ fm rather than the microphysics QCD length scale ~ 1 fm. In this scenario magnetic fields are generated by the peculiar motion of the electric dipoles which arises from the convective transfer of the latent heat released by the expanding bubble walls. The field strength at the QCDPT time has been estimated by Cheng and Olinto to be as large as

$$B_{QCD} \simeq 10^8 \text{ G} \tag{4.5}$$

on a maximal coherence length $l_{\rm coh} \simeq H^{-1}_{QCD}$. Once again, by assuming frozen-in evolution of the field, one can determine present time values:

$$B_0 \simeq 10^{-16} \text{ G} \qquad l_0 \simeq 1 \text{ pc} . \tag{4.6}$$

Using Eq.(4.4) Cheng and Olinto found that on the galactic length scale $B(\text{kpc}) \simeq 10^{-20}$ G which may have same phenomenological relevance if the galactic dynamo is very effective.

In a following work Sigl, Olinto and Jedamzik [142] investigated the possible role that hydrodynamic instabilities produced by the expanding bubble walls may have in generating strong magnetic fields . Although is not clear whether these instabilities can really develop during the QCDPT, Sigl at al. claimed that this phenomenon is not implausible for a reasonable choice of the QCDPT parameters. By taking into account the damping due to the finite viscosity and heat conductivity of the plasma, the authors of Ref.[142] showed that the instability may grow non-linearly producing turbulence on a scale of the order of the bubble size at the percolation time. As a consequence, a MHD dynamo may operate to amplify seed magnetic fields and equipartition of the magnetic field energy with the kinetic energy may be reached. If this is the case, magnetic fields of the order of 10^{-20} G may be obtained at the present time on a very large scale ~ 10 Mpc. Larger fields may be obtained by accounting for an inverse cascade (see Sec.1.4). In the most optimistic situation in which the magnetic field was produced having maximal helicity at the QCDPT and equipartion between magnetic and thermal energy was realized at that time, Eqs.(1.34,1.35) for the time evolution of the rms field strength and coherence length apply. By substituting in such equations the initial scale $l(T_i) \sim r_H(T_{QCD}) \sim 30$ km, and $B_{rmrms}(T_i) \sim 10^{17}$ G, one finds the present time values

$$B_{\rm rms}(T_0) \sim 10^{-9} \ \ {\rm G} \qquad L_{\rm coh} \sim 100 \ \ {\rm kpc} \ . \tag{4.7}$$

Remarkably, we see that in this optimistic case no dynamo amplification is required to explain galactic, and probably also cluster, magnetic fields.

4.3 Magnetic fields from the electroweak phase transition

4.3.1 Magnetic fields from a turbulent charge flow

Some of the ingredients which may give rise to magnetogenesis at the QCDPT may also be found at the electroweak phase transition (EWPT). As for the case of the QCDPT, magnetogenesis at the weak scale seems to require a first order transition. Although recent lattice computations performed in the framework of the standard electroweak theory [143] give a strong evidence against a first order transition, this remains a viable possibility if supersymmetric extension of the standard model are considered [144]. It is noticeable that a first order EWPT is also required for the successful realization of the electroweak baryogenesis scenario [145]. Indeed, as we shall see in the rest of this review, this is only one among several common aspects of baryogenesis and magnetogenesis.

According to Baym, Bödeker and McLerran [146] strong magnetic fields can be generated by a first order EWPT [1] via a dynamo mechanism. In this scenario seed fields are provided by random magnetic field fluctuations which are always present on a scale of the order of a thermal wavelength [2]. The amplification of such seed fields proceeds as follows. When the Universe

[1]At the time the paper by Baym et al. was written, a first order EWPT was thought to be compatible with the standard model. Therefore all computation in [146] were done in the framework of that model.

[2]It is worthwhile to observe here that thermal fluctuation in a dissipative plasma can actually produce stochastic magnetic fields on a scale larger than the thermal wavelength [147, 148]

supercooled below the critical temperature ($T_c \sim 100$ GeV) the Higgs field locally tunneled from the unbroken $\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ phase to the broken $U(1)_{em}$ phase. The tunneling gave rise to the formation of broken phase bubbles which then expanded by converting the false vacuum energy into kinetic energy. Although the bubble wall velocity is model dependent, one can find that for a wide range of the standard model parameters the expansion is subsonic (deflagration) which give rise to a supersonic shock wave ahead of the burning front. As the shock fronts collided turbulence should have formed in the cone associated with the bubble intersection. The Reynold number for the collision of two bubbles is

$$Re \sim \frac{v_{\mathrm{fluid}} R_{\mathrm{bubble}}}{\lambda}, \tag{4.8}$$

where $v_{\mathrm{fluid}} \sim v_{\mathrm{wall}} \sim 10^{-1}$ is the typical fluid velocity, R_{bubble} is the size of a bubble at the collision time and λ is the scattering length of excitations in the electroweak plasma. The typical size of a bubble after the phase transition is completed is in the range

$$R_{\mathrm{bubble}} \sim f_{\mathrm{b}} H_{\mathrm{ew}}^{-1} \tag{4.9}$$

where

$$H_{\mathrm{ew}}^{-1} \sim \frac{m_{\mathrm{Pl}}}{g_*^{1/2} T_{\mathrm{c}}^2} \sim 10 \text{ cm} \tag{4.10}$$

is the size of the event horizon at the electroweak scale, m_{Pl} is the Planck mass, $g_* \sim 10^2$ is the number of massless degrees of freedom in the matter, and the fractional size f_{b} is $\sim 10^{-2} - 10^{-3}$. The typical scattering length λ of excitations in the plasma is of order

$$\lambda \sim \frac{1}{T g_{\mathrm{ew}} \alpha_{\mathrm{w}}^2 |\ln \alpha_{\mathrm{w}}|}, \tag{4.11}$$

where α_{w} is the fine structure constant at the electroweak scale, and $g_{\mathrm{ew}} \sim g_*$ is the number of degrees of freedom that scatter by electroweak processes. By substituting these expressions Baym et al. found

$$Re \sim 10^{-3} \frac{m_{\mathrm{Pl}}}{T_{\mathrm{c}}} \alpha_{\mathrm{w}}^2 |\ln \alpha_{\mathrm{w}}| \sim 10^{12} \ . \tag{4.12}$$

Such a huge Reynolds number means that turbulence fully develops at all scales smaller than R_{bubble}. Since conductivity is expected to be quite large at that time [49], magnetic fields followed the fluid motion so that a strong magnetic turbulence should also have been produced. In such a situation it is known that the kinetic energy of the turbulent flow is equipartitioned with the magnetic field energy. Therefore

$$B^2(R_{\rm bubble}) \sim \epsilon(T_{\rm c}) v^2_{\rm fluid}, \tag{4.13}$$

where $\epsilon(T_{\rm c}) \sim g_* T_{\rm c}^4$ is the energy density of the electroweak plasma.

In order to estimate the magnetic field strength on scale larger than $R_{\rm bubble}$, Baym et al. treated the large scale field as a superposition of the field of dipoles with size $R_{\rm bubble}$. This is similar to what done by other authors [105, 140] (see what we wrote above for the QCDPT) apart for the fact that Baym et al. used a continuum approximation for the distribution of dipoles rather than to assume a random walk of the field lines. The density $\nu^i(\mathbf{r})$ of dipoles pointing in the i-th direction was assumed to be Gaussianly distributed. This implies the following correlation functions for the density of dipoles

$$\left\langle \nu^i(\mathbf{r})\nu^j(0)\right\rangle = \kappa \delta^{ij}\delta^{(3)}(\mathbf{r}) \tag{4.14}$$

and for the magnetic field

$$\langle \mathbf{B}(\mathbf{r})\cdot\mathbf{B}(\mathbf{0})\rangle \sim e^2\kappa \int d^3 r_{\rm d} \frac{1}{\mid \mathbf{r}-\mathbf{r}_{\rm d}\mid^3}\frac{1}{\mid \mathbf{r}_{\rm d}\mid^3}\ . \tag{4.15}$$

The logarithmic divergence of the integral in these regions is cut off by the size of the typical dipole, $f_{\rm b}H^{-1}_{\rm ew}$, so that for $r \gg f_{\rm b}H^{-1}_{\rm ew}$,

$$\langle \mathbf{B}(\mathbf{r})\cdot\mathbf{B}(\mathbf{0})\rangle \sim \frac{e^2\kappa}{r^3}\ln\left(\frac{H_{\rm ew}r}{f_{\rm b}}\right). \tag{4.16}$$

By using this expression and the equipartition relation (4.13) one finds that the strength of B^2 measured by averaging on a size scale R is

$$\left\langle B^2\right\rangle_R \sim v^2_{\rm fluid} g_* T_{\rm c}^4 \left(\frac{f_{\rm b}}{H_{\rm ew}R}\right)^3 \ln^2\left(\frac{H_{\rm ew}R}{f_{\rm b}}\right). \tag{4.17}$$

This result can be better expressed in terms of the ratio r of $\langle B^2\rangle$ to the energy ρ_γ in photons which is a constant during Universe expansion in the absence of flux diffusion. From Eqs.(4.17)one gets

$$r_R \sim v^2_{\rm fluid} f_{\rm b}^3 \left(\frac{\lambda_{\rm ew}}{R}\right)^3 \ln^2\left(\frac{R}{f_{\rm b}\lambda_{\rm ew}}\right), \tag{4.18}$$

where $\lambda_{\rm ew}$ is the Hubble radius at the electroweak phase transition (~ 1 cm) times the scale factor, $T_{\rm c}/T_\gamma$, where T_γ is the photon temperature.

From the previous results the authors of Ref.[146] estimated the average magnetic field strength at the present time. This is

$$B(l_{\rm diff}) \sim 10^{-7} \; - \; 10^{-9}\ {\rm G}, \tag{4.19}$$

where $l_{\rm diff} \sim 10$ AU is present time diffusion length, and

$$B(l_{\rm gal}) \sim 10^{-17} \; - \; 10^{-20}\ {\rm G}, \tag{4.20}$$

on the galactic scale $l_{\rm gal} \sim 10^9$ AU.

4.3.2 Magnetic fields from Higgs field equilibration

In the previous section we have seen that, concerning the generation of magnetic fields, the QCDPT and the EWPT share several common aspects. However, there is one important aspect which makes the EWPT much more interesting than the QCDPT. In fact, at the electroweak scale the electromagnetic field is directly influenced by the dynamics of the Higgs field which drives the EWPT.

To start with we remind that, as a consequence of the Weinberg-Salam theory, before the EWPT is not even possible to define the electromagnetic field, and that this operation remains highly non-trivial until the transition is completed. In a sense, we can say that the electromagnetic field was "born" during the EWPT. The main problem in the definition of the electromagnetic field at the weak scale is the breaking of the translational invariance: the Higgs field module and its $SU(2)$ and $U_Y(1)$ phases take different values in different positions. This is either a consequence of the presence of thermal fluctuations, which close to T_c are locally able to break/restore the $SU(2)\times U_Y(1)$ symmetry or of the presence of large stable domains, or bubbles, where the broken symmetry has settled.

The first generalized definition of the electromagnetic field in the presence of a non-trivial Higgs background was given by t'Hooft [149] in the seminal paper where he introduced magnetic monopoles in a $SO(3)$ Georgi-Glashow model. t'Hooft definition is the following

$$\mathcal{F}^{\rm em}_{\mu\nu} \equiv \hat{\phi}^a G^a_{\mu\nu} + g^{-1}\epsilon^{abc}\hat{\phi}^a(D_\mu\hat{\phi})^b(D_\nu\hat{\phi})^c \; . \tag{4.21}$$

In the above $G^a_{\mu\nu} \equiv \partial W^a_\mu - \partial W^a_\nu$, where

$$\hat{\phi}^a \equiv \frac{\Phi^\dagger \tau^a \Phi}{\Phi^\dagger \Phi} \tag{4.22}$$

(τ^a are the Pauli matrices) is a unit isovector which defines the "direction" of the Higgs field in the $SO(3)$ isospace (which coincides with $SU(2)$) and $(D_\mu \hat{\phi})^a = \partial_\mu \hat{\phi}^a + g\epsilon^{abc} W^b_\mu \hat{\phi}^c$, where W^b_μ are the gauge fields components in the adjoint representation. The nice features of the definition (4.21) are that it is gauge-invariant and it reduces to the standard definition of the electromagnetic field tensor if a gauge rotation can be performed so to have $\hat{\phi}^a = -\delta^{a3}$ (unitary gauge). In some models, like that considered by t'Hooft, a topological obstruction may prevent this operation to be possible everywhere. In this case singular points (monopoles) or lines (strings) where $\phi^a = 0$ appear which become the source of magnetic fields. t'Hooft result provides an existence proof of magnetic fields produced by non-trivial vacuum configurations.

The Weinberg-Salam theory, which is based on the $SU(2) \times U_Y(1)$ group representation, does not predict topologically stable field configurations. We will see, however, that vacuum non-topological configurations possibly produced during the EWPT can still be the source of magnetic fields.

A possible generalization of the definition (4.21) for the Weinberg-Salam model was given by Vachaspati [106]. It is

$$\begin{aligned} F^{\rm em}_{\mu\nu} &\equiv -\sin\theta_W \hat{\phi}^a F^a_{\mu\nu} + \cos\theta_W F^Y_{\mu\nu} \\ &- i\frac{\sin\theta_W}{g}\frac{2}{\Phi^\dagger\Phi}\left[(D_\mu\Phi)^\dagger D_\nu\Phi - (D_\nu\Phi)^\dagger D_\mu\Phi\right], \end{aligned} \tag{4.23}$$

where $D_\mu = \partial_\mu - i\frac{g}{2}\tau^a W^a_\mu - i\frac{g'}{2}Y_\mu$.

This expression was used by Vachaspati to argue that magnetic fields should have been produced during the EWPT. Synthetically, Vachaspati argument is the following. It is known that well below the EWPT critical temperature T_c the minimum energy state of the Universe corresponds to a spatially homogeneous vacuum in which Φ is covariantly constant, i.e. $D_\nu\Phi = D_\mu\hat{\phi}^a = 0$. However, during the EWPT, and immediately after it, thermal fluctuations give rise to a finite correlation length $\xi \sim (eT_c)^{-1}$. Therefore, there are spatial variations both in the Higgs field module $|\Phi|$ and in its $SU(2)$ and $U(1)_Y$ phases which take random values in uncorrelated regions [3]. It was noted by Davidson [150] that gradients in the radial part

[3]Vachaspati [106] did also consider Higgs field gradients produced by the presence of

of the Higgs field cannot contribute to the production of magnetic fields as this component is electrically neutral. While this consideration is certainly correct, it does not imply the failure of Vachaspati argument. In fact, the role played by the spatial variations of the $SU(2)$ and $U(1)_Y$ "phases" of the the Higgs field cannot be disregarded. It is worthwhile to observe that gradients of these phases are not a mere gauge artifact as they correspond to a nonvanishing kinetic term in the Lagrangian. Of course one can always rotate Higgs fields phases into gauge boson degrees of freedom (see below) but this operation does not change $F^{\rm em}_{\mu\nu}$ which is a gauge-invariant quantity. The contribution to the electromagnetic field produced by gradients of $\hat{\phi}^a$ can be readily determined by writing the Maxwell equations in the presence of an inhomogeneous Higgs background [151]

$$\begin{aligned} \partial^\mu F^{\rm em}_{\mu\nu} = & - \sin\theta_W \left\{ D^\mu \hat{\phi}^a F^a_{\mu\nu} \right. \\ & + \left. \frac{i}{g}\partial^\mu \left[\frac{4}{\Phi^\dagger\Phi} \left((D_\mu\Phi)^\dagger D_\nu\Phi - D_\mu\Phi\,(D_\nu\Phi)^\dagger \right) \right] \right\} . \end{aligned} \tag{4.24}$$

Even neglecting the second term on the righthand side of Eq.(4.24), which depends on the definition of $F^{\rm em}_{\mu\nu}$ in a Higgs inhomogeneous background (see below), it is evident that a nonvanishing contribution to the electric 4-current arises from the covariant derivative of $\hat{\phi}^a$. The physical meaning of this contribution may look more clear to the reader if we write Eq.(4.24) in the unitary gauge

$$\begin{aligned} \partial^\mu F^{\rm em}_{\mu\nu} = & +ie\left[W^{\mu\dagger}\left(D_\nu W_\mu\right) - W^\mu \left(D_\nu W_\mu\right)^\dagger \right] \\ & - ie\left[W^{\mu\dagger}\left(D_\mu W_\nu\right) - W^\mu \left(D_\mu W_\nu\right)^\dagger \right] \\ & - ie\partial^\mu \left(W^\dagger_\mu W_\nu - W_\mu W^\dagger_\nu \right) . \end{aligned} \tag{4.25}$$

Not surprisingly, we see that the electric currents produced by Higgs field equilibration after the EWPT are nothing but W boson currents.

Since, on dimensional grounds, $D_\nu\Phi \sim v/\xi$ where v is the Higgs field vacuum expectation value, Vachaspati concluded that magnetic fields (electric fields were supposed to be screened by the plasma) should have been produced at the EWPT with strength

$$B \sim \sin\theta_W g T_c^2 \approx 10^{23} \ \ {\rm G} . \tag{4.26}$$

the cosmological horizon. However, since the Hubble radius at the EWPT is of the order of 1 cm whereas $\xi \sim (eT_c)^{-1} \sim 10^{-16}$ cm, it is easy to realize that magnetic fields possibly produced by the presence of the cosmological horizon are phenomenologically irrelevant.

Of course these fields live on a very small scale of the order of ξ and in order to determine fields on a larger scale Vachaspati claimed that a suitable average has to be performed (see return on this issue below in this section).

Before discussing averages, however, let us try to understand better the nature of the magnetic fields which may have been produced by the Vachaspati mechanism. We notice that Vachaspati's derivation does not seem to invoke any out-of-equilibrium process and indeed the reader may wonder what is the role played by the phase transition in the magnetogenesis. Moreover, magnetic fields are produced anyway on a scale $(eT)^{-1}$ by thermal fluctuations of the gauge fields so that it is unclear what is the difference between magnetic fields produced by the Higgs fields equilibration and these more conventional fields. In our opinion, although Vachaspati's argument is basically correct its formulation was probably oversimplified. Indeed, several works showed that in order to reach a complete understanding of this physical effect a more careful study of the dynamics of the phase transition is called for. We shall now review these works starting from the case of a first order phase transition.

The case of a first order EWPT

Before discussing the $SU(2) \times U(1)$ case we cannot overlook some important work which was previously done about phase equilibration during bubble collision in the framework of more simple models. In the context of a $U(1)$ Abelian gauge symmetry, Kibble and Vilenkin [152] showed that the process of phase equilibration during bubble collisions give rise to relevant physical effects. The main tool developed by Kibble and Vilenkin to investigate this kind of processes is the, so-called, *gauge-invariant phase difference* defined by

$$\Delta\theta = \int_A^B dx^k \ D_k\theta \ , \qquad (4.27)$$

where θ is the $U(1)$ Higgs field phase and $D_\mu\theta \equiv \partial_\mu\theta + eA_\mu$ is the phase covariant derivative. A and B are points taken in the bubble interiors and $k = 1, 2, 3$. $\Delta\theta$ obeys the Klein-Gordon equation

$$(\partial^2 + m^2)\Delta\theta = 0 \qquad (4.28)$$

where $m = ev$ is the gauge boson mass. Kibble and Vilenkin assumed that during the collision the radial mode of the Higgs field is strongly damped so that it rapidly settles to its expectation value v everywhere. One can choose a frame of reference in which the bubbles are nucleated simultaneously with centers at $(t, x, y, z) = (0, 0, 0, \pm R_c)$. In this frame, the bubbles have equal initial radius $R_i = R_0$. Their first collision occurs at $(t_c, 0, 0, 0)$ when their radii are R_c and $t_c = \sqrt{R_c^2 - R_0^2}$. Given the symmetry of the problem about the axis joining the nucleation centers (z-axis), the most natural gauge is the axial gauge. In this gauge

$$\theta(x) = \theta(\tau, z), \quad A^\alpha(x) = x^\alpha a(\tau, z) \ , \tag{4.29}$$

where $\alpha = 0, 1, 2$ and $\tau^2 = t^2 - x^2 - y^2$. The condition $\theta_a(\tau, 0) = 0$ fixes the gauge completely. At the point of first contact $z = 0$, $\tau = t_c$ the Higgs field phase was assumed to change from θ_0 to $-\theta_0$ going from a bubble into the other. This constitutes the initial condition of the problem. The following evolution of θ is determined by the Maxwell equation

$$\partial^\nu F_{\mu\nu} = j_\mu = -ev^2 \ D_\mu \theta \tag{4.30}$$

and the Klein-Gordon equation which splits into

$$\begin{aligned} \partial_\tau^2 \theta_a + \frac{2}{\tau}\partial_\tau \theta - \partial_z^2 \theta + m^2 \theta &= 0 \ , & (4.31) \\ \partial_\tau^2 a + \frac{4}{\tau}\partial_\tau a - \partial_z^2 a + m^2 a &= 0 \ . & (4.32) \end{aligned}$$

The solution of the linearized equations (4.31) and (4.32) for $\tau > t_c$ then becomes

$$\begin{aligned} \theta_a(\tau, z) &= \frac{\theta_0 t_c}{\pi\tau} \int_{-\infty}^{\infty} \frac{dk}{k} \sin kz \left(\cos\omega(\tau - t_c) + \frac{1}{\omega t_c} \sin\omega(\tau - t_c) \right) & (4.33) \\ a(\tau, z) &= \frac{\theta_0 m^2 t_c}{\pi e \tau^3} \int_{-\infty}^{\infty} \frac{dk}{k} \sin kz \left[-\frac{\tau - t_c}{\omega^2 t_c} \cos\omega(\tau - t_c) \right. \\ &\quad \left. + \left(\frac{\tau}{\omega} + \frac{1}{\omega^3 t_c} \right) \sin\omega(\tau - t_c) \right] \ , & (4.34) \end{aligned}$$

where $\omega^2 = k^2 + m^2$. The gauge-invariant phase difference is deduced by the asymptotic behavior at $z \to \pm\infty$

$$\begin{aligned} \Delta\theta &= \theta_a(t, 0, 0, +\infty) - \theta_a(t, 0, 0, -\infty) & (4.35) \\ &= \frac{2\theta_0 t_c}{t} \left(\cos m(t - t_c) + \frac{1}{m t_c} \sin m(t - t_c) \right) \ . \end{aligned}$$

Thus, phase equilibration occurs with a time scale t_c determined by the bubble size, with superimposed oscillations with frequency given by the gauge-field mass. As we see from Eq.(4.34) phase oscillations come together with oscillations of the gauge field. It follows from Eq.(4.30) that these oscillations give rise to an "electric" current. This current will source an "electromagnetic" field strength $F_{\mu\nu}$ [4]. Because of the symmetry of the problem the only nonvanishing component of $F_{\mu\nu}$ is

$$F^{\alpha z} = x^{\alpha} \partial_z a(\tau, z) \ . \tag{4.36}$$

Therefore, we have an azimuthal magnetic field $B^{\varphi} = F^{z\rho} = \rho \partial_z a$ and a longitudinal electric field $E^z = F^{0z} = -t\partial_z a = -(t/\rho) B^{\varphi}(\tau, z)$, where we have used cylindrical coordinates (ρ, ϕ). We see that phase equilibration during bubble collision indeed produces some real physical effects.

Kibble and Vilenkin did also consider the role of electric dissipation. They showed that a finite value of the electric conductivity σ gives rise to a damping in the "electric" current which turns into a damping for the phase equilibration. They found

$$\Delta\theta(t) = 2\theta_0 e^{-\sigma t/2} \left(\cos mt + \frac{\sigma}{2m} \sin mt \right) \tag{4.37}$$

for small values of σ, and

$$\Delta\theta(t) = 2\theta_0 \exp\left(-m^2 t/\sigma\right) \tag{4.38}$$

in the opposite case. The dissipation time scale is typically much smaller than the time which is required for two colliding bubble to merge completely. Therefore the gauge-invariant phase difference settles rapidly to zero in the overlapping region of the two bubbles and in its neighborhood. It is interesting to compute the line integral of $D_k\theta$ over the path ABCD represented in the Fig.4.1. From the previous considerations it follows that $\Delta\theta_{AB} = 0$, $\Delta\theta_{AD} = \Delta\theta_{BC} = 0$ and $\Delta\theta_{DC} = 2\theta_0$. It is understood that in order for the integral to be meaningful, the vacuum expectation value of the Higgs field has to remain nonzero in the collision region and around it, so that the phase θ remains well defined and interpolates smoothly between its values inside the bubbles. Under these hypothesis we have

$$\oint_{ABCD} D_k\theta dx^k = 2\theta_0 \ . \tag{4.39}$$

[4] It is understood that since the toy model considered by Kibble e Vilenkin is not $SU(2) \times U(1)_Y$, $F_{\mu\nu}$ is not the physical electromagnetic field strength.

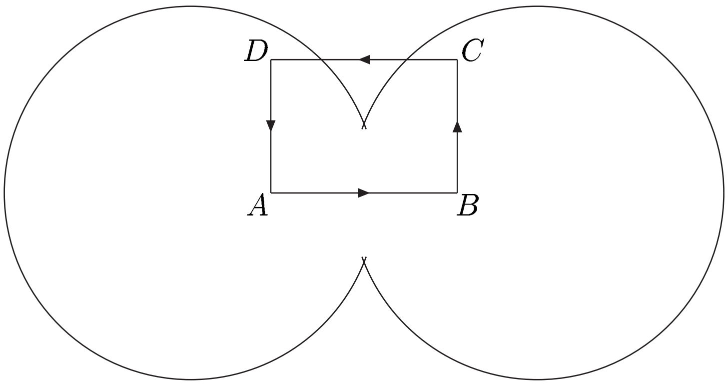

Figure 4.1: Two colliding bubbles. It is showed the closed path along which the displacement of the gauge invariant phase difference is computed. (From Ref.[152]).

The physical meaning of this quantity is recognizable at a glance in the unitary gauge, in which each $\Delta\theta$ is given by a line integral of the vector potential $\mathbf{A}$. We see that the gauge-invariant phase difference computed along the loop is nothing but the magnetic flux trough the loop itself

$$\phi(B) = \oint_{ABCD} A_k \theta dx^k = \frac{1}{e} \oint_{ABCD} D_k \theta dx^k = \frac{2\theta_0}{e} \ . \tag{4.40}$$

In other words, phase equilibration give rise to a ring of magnetic flux near the circle on which bubble walls intersect. If the initial phase difference between the two bubbles is 2π, the total flux trapped in the ring is exactly one flux quantum, $2\pi/e$.

Kibble and Vilenkin did also consider the case in which three bubbles collide. They argued that in this case the formation of a string, in which interior symmetry is restored, is possible. Whether or not this happens is determined by the net phase variation along a closed path going through the three bubbles. The string forms if this quantity is larger than 2π. According to Kibble and Vilenkin strings cannot be produced by two bubble collisions because, for energetic reasons, the system will tend to choose the shorter of the two paths between the bubble phases so that a phase displacement $\geq 2\pi$ can never be obtained. This argument, which was first used by Kibble [153] for the study of defect formation, is often called the "geodesic rule".

The work of Kibble and Vilenkin was reconsidered by Copeland and Saffin [154] and more recently by Copeland, Saffin and Törnkvist [155] who showed that during bubble collision the dynamics of the radial mode of the Higgs field cannot really be disregarded. In fact, violent fluctuations in the modulus of the Higgs field take place and cause symmetries to be restored locally, allowing the phase to "slip" by an integer multiple of 2π violating the geodesic rule. Therefore strings, which carry a magnetic flux, can be produced also by the collision of only two bubbles. Saffin and Copeland [156] went a step further by considering phase equilibration in the $SU(2)\times U(1)$ case, namely the electroweak case. They showed that for some particular initial conditions the $SU(2)\times U(1)$ Lagrangian is equivalent to a $U(1)$ Lagrangian so that part of Kibble and Vilenkin [152] considerations can be applied. The violation of the geodesic rule allows the formation of vortex configurations of the gauge fields. Saffin and Copeland argued that these configurations are related to the Nielsen-Olesen vortices [157]. Indeed, it is know that such a kind of non-perturbative solutions are allowed by the Weinberg-Salam model [158] (for a comprehensive review on electroweak strings see Ref.[159]). Although electroweak string are not topologically stable, numerical simulations performed in Ref.[156] show that in presence of small perturbations the vortices survives on times comparable to the time required for bubble to merge completely.

The generation of magnetic fields in the $SU(2)\times U(1)_Y$ case was not considered in the work by Saffin and Copeland. This issue was the subject of a following paper by Grasso and Riotto [151]. The authors of Ref.[151] studied the dynamics of the gauge fields starting from the following initial Higgs field configuration

$$\Phi_{\rm in}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \rho(\mathbf{x}) \end{pmatrix} + \frac{1}{\sqrt{2}}\,\exp\left(-i\frac{\theta_0}{2}n^a\tau^a\right)\begin{pmatrix} 0 \\ \rho(\mathbf{x}-\mathbf{b})e^{i\varphi_0} \end{pmatrix} \tag{4.41}$$

which represents the superposition of the Higgs fields of two bubbles which are separated by a distance b. In the above n^a is a unit vector in the $SU(2)$ isospace and τ^a are the Pauli matrices. The phases and the orientation of the Higgs field were chosen to be uniform across any single bubble. It was assumed that Eq.(4.41) holds until the two bubble collide ($t=0$). Since $n^a\tau^a$ is the only Lie-algebra direction which is involved before the collision, one can write the initial Higgs field configuration in the form [156]

$$\Phi_{\rm in}(\mathbf{x}) = \frac{1}{\sqrt{2}}\exp\left(-i\frac{\theta(\mathbf{x})}{2}n^a\tau^a\right)\begin{pmatrix} 0 \\ \rho(\mathbf{x})e^{i\varphi(\mathbf{x})} \end{pmatrix}. \tag{4.42}$$

In order to disentangle the peculiar role played by the Higgs field phases, the initial gauge fields W_μ^a and their derivatives were assumed to be zero at $t = 0$. This condition is of course gauge dependent and should be interpreted as a gauge choice. It is convenient to write the equation of motion for the gauge fields in the adjoint representation. For the $SU(2)$ gauge fields we have

$$D^\mu F_{\mu\nu}^a = g|\rho|^2 \ \epsilon^{abc} D_\nu \hat{\phi}^b \hat{\phi}^c \tag{4.43}$$

where the isovector $\hat{\phi}^a$ has been defined in Eq.(4.22). Under the assumptions mentioned in the above, at $t = 0$, this equation reads

$$\partial^\mu F_{\mu\nu}^a = -g|\rho|^2 \partial_\nu \theta(x) \left(n^a - n^c \hat{\phi}^a \hat{\phi}^c \right) \ . \tag{4.44}$$

In general, the unit isovector $\hat{\phi}^a$ can be decomposed into

$$\hat{\phi} = \cos\theta \ \hat{\phi}_0 + \sin\theta \ \hat{n} \times \hat{\phi}_0 + 2\sin^2\frac{\theta}{2} \ \left(\hat{n} \cdot \hat{\phi}_0 \right) \ \hat{n} \tag{4.45}$$

where $\hat{\phi}_0^T \equiv -(0,0,1)$. It is straightforward to verify that in the unitary gauge, $\hat{\phi}$ reduces to $\hat{\phi}_0$. The relevant point in Eq.(4.42) is that the versor $\hat{n}$, about which it is performed the $SU(2)$ gauge rotation, does not depend on the space coordinates. Therefore, without loosing generality, we have the freedom to choose $\hat{n}$ to be everywhere perpendicular to $\hat{\phi}_0$. In other words, $\hat{\phi}$ can be everywhere obtained by rotating $\hat{\phi}_0$ by an angle θ in the plane identified by $\hat{n}$ and $\hat{\phi}_0$. Formally, $\hat{\Phi} = \cos\theta \hat{\phi}_0 + \sin\theta \ \hat{n} \times \hat{\phi}_0$, which clearly describes a simple $U(1)$ transformation. In fact, since it is evident that the condition $\hat{n} \perp \hat{\phi}_0$ also implies $\hat{n} \perp \hat{\phi}$, the equation of motion (4.44) becomes

$$\partial^\mu F_{\mu\nu}^a = -g|\rho|^2 \partial_\nu \theta(x) n^a \ . \tag{4.46}$$

As expected, we see that only the gauge field component along the direction $\hat{n}$, namely $A_\mu = n^a W_\mu^a$, that has some initial dynamics which is created by a nonvanishing gradient of the phase between the two domains. When we generalize this result to the full $SU(2) \times U(1)_Y$ gauge structure, an extra generator, namely the hypercharge, comes-in. Therefore in this case is not any more possible to choose an arbitrary direction for the unit vector $\hat{n}$ since different orientations of the unit vector $\hat{n}$ with respect to $\hat{\phi}_0$ correspond to different physical situations. We can still consider the case in which $\hat{n}$ is parallel to $\hat{\phi}_0$ but we should take in mind that this is not the only possibility. In this case we have

$$\partial^\mu F^3_{\mu\nu} = \frac{g}{2}\rho^2(x)\left(\partial_\nu\theta + \partial_\nu\varphi\right), \tag{4.47}$$

$$\partial^\mu F^Y_{\mu\nu} = -\frac{g'}{2}\rho^2(x)\left(\partial_\nu\theta + \partial_\nu\varphi\right) \tag{4.48}$$

where g and g' are respectively the $SU(2)$ and $U(1)_Y$ gauge coupling constants. It is noticeable that in this case the charged gauge fields are not excited by the phase gradients at the time when bubble first collide. We can combine Eqs.(4.47) and (4.48) to obtain the equation of motion for the Z-boson field

$$\partial^\mu F^Z_{\mu\nu} = \frac{\sqrt{g^2+g'^2}}{2}\rho^2(x)\left(\partial_\nu\theta + \partial_\nu\varphi\right). \tag{4.49}$$

This equation tells us that a gradient in the phases of the Higgs field gives rise to a nontrivial dynamics of the Z-field with an effective gauge coupling constant $\sqrt{g^2+g'^2}$. We see that the equilibration of the phase $(\theta+\varphi)$ can be now treated in analogy to the $U(1)$ toy model studied by Kibble and Vilenkin [152], the role of the $U(1)$ "electromagnetic" field being now played by the Z-field. However, differently from Ref.[152] the authors of Ref.[151] left the Higgs field modulus free to change in space. Therefore, the equation of motion of $\rho(x)$ has to be added to (4.49). Assuming the charged gauge field does not evolve significantly, the complete set of equations of motion that we can write at finite, though small, time after the bubbles first contact, is

$$\begin{aligned} \partial^\mu F^Z_{\mu\nu} &= \frac{g}{2\cos\theta_W}\rho^2(x)\left(\partial_\nu\varphi + \frac{g}{2\cos\theta_W}Z_\nu\right), \\ d^\mu d_\mu\left(\rho(x)\mathrm{e}^{i\frac{\varphi}{2}}\right) &+ 2\lambda\left(\rho^2(x) - \frac{1}{2}\eta^2\right)\rho(x)\mathrm{e}^{i\frac{\varphi}{2}} = 0, \end{aligned} \tag{4.50}$$

where $d_\mu = \partial_\mu + i\frac{g}{2\cos\theta_W}Z_\mu$, η is the vacuum expectation value of Φ and λ is the quartic coupling. Note that, in analogy with [152], a gauge invariant phase difference can be introduced by making use of the covariant derivative d_μ. Equations (4.50) are the Nielsen-Olesen equations of motion [157]. Their solution describes a Z-vortex where $\rho = 0$ at its core [160]. The geometry of the problem implies that the vortex is closed, forming a ring which axis coincide with the conjunction of bubble centers. This result provides further support to the possibility that electroweak strings are produced during the EWPT.

In principle, in order to determine the magnetic field produced during the process that we illustrated in the above, we need a gauge-invariant definition of the electromagnetic field strength in the presence of the non-trivial Higgs background. We know however that such definition is not unique [161]. For example, the authors of Ref.[151] used the definition given in Eq.(4.23) to find that the electric current is

$$\partial^\mu F^{\rm em}_{\mu\nu} = 2\tan\theta_W \partial^\mu \left(Z_\mu \partial_\nu \ln\rho(x) - Z_\nu \partial_\mu \ln\rho(x)\right) \tag{4.51}$$

whereas other authors [162], using the definition

$$\mathcal{F}^{\rm em}_{\mu\nu} \equiv -\sin\theta_W \hat\phi^a F^a_{\mu\nu} + \cos\theta_W F^Y_{\mu\nu} + \frac{\sin\theta_W}{g}\epsilon^{abc}\hat\phi^a (D_\mu\hat\phi)^b (D_\nu\hat\phi)^c\ , \tag{4.52}$$

found no electric current, hence no magnetic field, at all. We have to observe, however, that the choice between these, as others, gauge invariant definitions is more a matter of taste than physics. Different definitions just give the same name to different combinations of the gauge fields. The important requirement which acceptable definitions of the electromagnetic field have to fulfill is that they have to reproduce the standard definition in the broken phase with a uniform Higgs background. This requirement is fulfilled by both the definitions used in the Refs.[151] and [162]. In our opinion, it is not really meaningful to ask what is the electromagnetic field inside, or very close to, the electroweak strings. The physically relevant question is what are the electromagnetic relics of the electroweak strings once the EWPT is concluded.

One important point to take in mind is that electroweak strings are not topologically stable (see [159] and references therein) and that, for the physical value of the Weinberg angle, they rapidly decay after their formation. Depending on the nature of the decay process two scenarios are possible. According to Vachaspati [163] long strings should decay in short segments of length $\sim m_W^{-1}$. Since the Z-string carry a flux of Z-magnetic flux in its interior

$$\phi_Z = \frac{4\pi}{\alpha} = \frac{4\pi}{e}\sin\theta_W\ \cos\theta_W\ . \tag{4.53}$$

and the Z gauge field is a linear superposition of the W^3 and Y fields then, when the string terminates, the Y flux cannot terminate because it is a $U(1)$

gauge field and the Y magnetic field is divergenceless. Therefore some field must continue even beyond the end of the string. This has to be the massless field of the theory, that is, the electromagnetic field. In some sense, a finite segment of Z-string terminates on magnetic monopoles [158]. The magnetic flux emanating from a monopole is:

$$\phi_B = \frac{4\pi}{\alpha} \tan\theta_W = \frac{4\pi}{e} \sin^2\theta_W \ . \tag{4.54}$$

This flux may remain frozen-in the surrounding plasma and become a seed for cosmological magnetic fields.

Another possibility is that Z-strings decay by the formation of a W-condensate in their cores. In fact, it was showed by Perkins [164] that while electroweak symmetry restoration in the core of the string reduces m_W, the magnetic field via its coupling to the anomalous magnetic moment of the W-field, causes, for $eB > m_W^2$, the formation of a condensate of the W-fields. Such a process is based on the Ambjorn-Olesen instability which will be discussed in some detail in the Chap.5 of this review. As noted in [151] the presence of an inhomogeneous W-condensate produced by string decay gives rise to electric currents which may sustain magnetic fields even after the Z string has disappeared. The formation of a W-condensate by strong magnetic fields at the EWPT time, was also considered by Olesen [165].

We can now wonder what is the predicted strength of the magnetic fields at the end of the EWPT. An attempt to answer to this question has been done by Ahonen and Enqvist [166] (see also Ref.[167]) where the formation of ring-like magnetic fields in collisions of bubbles of broken phase in an Abelian Higgs model were inspected. Under the assumption that magnetic fields are generated by a process that resembles the Kibble and Vilenkin [152] mechanism, it was concluded that a magnetic field of the order of $B \simeq 2\times 10^{20}$ G with a coherence length of about 10^2 GeV^{-1} may be originated. Assuming turbulent enhancement the authors of Ref.[166] of the field by inverse cascade [51], a root-mean-square value of the magnetic field $B_{\rm rms} \simeq 10^{-21}$ G on a comoving scale of 10 Mpc might be present today. Although our previous considerations give some partial support to the scenario advocated in [166] we have to stress, however, that only in some restricted cases it is possible to reduce the dynamics of the system to the dynamics of a simple $U(1)$ Abelian group. Furthermore, once Z-vortices are formed the non-Abelian nature of the electroweak theory shows due to the back-reaction of the magnetic field on the charged gauge bosons and it is not evident that the same numerical values obtained in [166] will be obtained in the case of the EWPT.

However the most serious problem with the kind of scenario discussed in this section comes form the fact that, within the framework of the standard model, a first order EWPT seems to be incompatible with the Higgs mass experimental lower limit [143]. Although some parameter choice of the minimal supersymmetric standard model (MSSM) may still allow a first order transition [144], which may give rise to magnetic fields in a way similar to that discussed in the above, we think it is worthwhile to keep an open mind and consider what may happen in the case of a second order transition or even in the case of a cross over.

The case of a second order EWPT

As we discussed in the first part of this section, magnetic fields generation by Higgs field equilibration share several common aspects with the formation of topological defects in the early Universe. This analogy holds, and it is even more evident, in the case of a second order transition. The theory of defect formation during a second order phase transition was developed in a seminal paper by Kibble [153]. We shortly review some relevant aspects of the Kibble mechanism. We start from the Universe being in the unbroken phase of a given symmetry group G. As the Universe cools and approach the critical temperature T_c protodomains are formed by thermal fluctuations where the vacuum is in one of the degenerate, classically equivalent, broken symmetry vacuum states. Let M be the manifold of the broken symmetry degenerate vacua. The protodomains size is determined by the Higgs field correlation function. Protodomains become stable to thermal fluctuations when their free energy becomes larger than the temperature. The temperature at which this happens is usually named Ginsburg temperature T_G. Below T_G stable domains are formed which, in the case of a topologically nontrivial manifold M, give rise to defect production. Rather, if M is topologically trivial, phase equilibration will continue until the Higgs field is uniform everywhere. This is the case of the Weinberg-Salam model, as well as of its minimal supersymmetrical extension.

Higgs phase equilibration, which occurs when stable domains merge, gives rise to magnetic fields in a way similar to that described by Vachaspati [106] (see the beginning of this section). One should keep in mind, however, that as a matter of principle, the domain size, which determine the Higgs field gradient, is different from the correlation length at the critical temperature

[151]. At the time when stable domains form, their size is given by the correlation length in the broken phase at the Ginsburg temperature. This temperature was computed, in the case of the EWPT, by the authors of Ref.[151] by comparing the expansion rate of the Universe with the nucleation rate per unit volume of sub-critical bubbles of symmetric phase (with size equal to the correlation length in the broken phase) given by

$$\Gamma_{\rm ub} = \frac{1}{\ell_{\rm b}^4}\, {\rm e}^{-S_3^{\rm ub}/T}, \tag{4.55}$$

where $\ell_{\rm b}$ is the correlation length in the broken phase. $S_3^{\rm ub}$ is the high temperature limit of the Euclidean action (see e.g. Ref.[168]). It was shown that for the EWPT the Ginsburg temperature is very close to the critical temperature, $T_G = T_c$ within a few percent. The corresponding size of a broken phase domain is determined by the correlation length in the broken phase at $T = T_G$

$$\frac{1}{\ell(T_G)_{\rm b}^2} = V''\left(\langle\phi(T_G)\rangle, T_G\right) \tag{4.56}$$

where $V(\phi, T)$ is the effective Higgs potential. $\ell(T_G)_{\rm b}^2$ is weakly dependent on M_H, $\ell_{\rm b}(T_G) \simeq 11/T_G$ for $M_H = 100$ GeV and $\ell_{\rm b}(T_G) \simeq 10/T_G$ for $M_H = 200$ GeV. Using this result and Eq.(4.23) the authors of Ref.[151] estimated the magnetic field strength at the end of the EWPT to be of order of

$$B_\ell \sim 4e^{-1}\sin^2\theta_W \ell_{\rm b}^2(T_G) \sim 10^{22}\ {\rm G}\ , \tag{4.57}$$

on a length scale $\ell_{\rm b}(T_G)$.

Although it was showed by Martin and Davis [169] that magnetic fields produced on such a scale may be stable against thermal fluctuations, it is clear that magnetic fields of phenomenological interest live on scales much larger than $\ell_{\rm b}(T_G)$. Therefore, some kind of average is required. We are ready to return to the discussion of the Vachaspati mechanism for magnetic field generation [106]. Let us suppose we are interested in the magnetic field on a scale $L = N\ell$. Vachaspati argued that, since the Higgs field is uncorrelated on scales larger than ℓ, its gradient executes a random walk as we move along a line crossing N domains. Therefore, the average of the gradient $D_\mu\Phi$ over this path should scale as $\sqrt{N}$. Since the magnetic field is proportional to the product of two covariant derivatives, see Eq.(4.23), Vachaspati concluded that it scales as $1/N$. This conclusion, however, overlooks the difference between $\langle D_\mu\Phi^\dagger\rangle\langle D_\mu\Phi\rangle$ and $\langle D_\mu\Phi^\dagger D_\mu\Phi\rangle$. This point was noticed by Enqvist and Olesen [107] (see also Ref.[109]) who produced a different estimate for the average magnetic field , $\langle B\rangle_{\rm rms,\ L} \equiv B(L) \sim B_\ell/\sqrt{N}$. Neglecting the possible role of the magnetic helicity (see the next section) and of possible related effects, e.g. inverse cascade, and using Eq.(4.57), the line-averaged field today on a scale $L \sim 1$ Mpc ($N \sim 10^{25}$) is found to be of the order $B_0(1{\rm Mpc}) \sim 10^{-21}$ G.

Another important point of this kind of scenario (for the reasons which will become clear in the next section) is that it naturally gives rise to a nonvanishing vorticity. This point can be understood by the analogy with the process which lead to the formation of superfluid circulation in a Bose-Einstein fluid which is rapidly taken below the critical point by a pressure quench [170]. Consider a circular closed path through the superfluid of length $C = 2\pi R$. This path will cross $N \simeq C/\ell$ domains, where ℓ is the characteristic size of a single domain. Assuming that the phase θ of the condensate wave function is uncorrelated in each of the N domains (random-walk hypothesis) the typical mismatch of θ is given by:

$$\delta\theta = \int_C \nabla\theta \cdot \mathbf{ds} \sim \sqrt{N} \tag{4.58}$$

where $\nabla\theta$ is the phase gradient across two adjacent domains and $\mathbf{ds}$ is the line element along the circumference. It is well known (see e.g. [171]) that from the Schrödinger equation it follows that the velocity of a superfluid is given by the gradient of the phase trough the relation $\mathbf{v}_s = (\hbar/m)\nabla\theta$, therefore (4.58) implies

$$v_s = (\hbar/m)\frac{1}{d}\frac{1}{\sqrt{N}}\ . \tag{4.59}$$

It was argued by Zurek [170] that this phenomenon can effectively simulate the formation of defects in the early Universe. As we discussed in the previous section, although the standard model does not allows topological defects, embedded defects, namely electroweak strings, may be produced through a similar mechanism. Indeed a close analogy was showed to exist [172] between the EWPT and the ^3He superfluid transition where formation of vortices is experimentally observed. This hypothesis received further support by some recent lattice simulations which showed evidence for the formation of a cluster of Z-strings just above the cross-over temperature [173] in the case of a $3D$ $SU(2)$ Higgs model. Electroweak strings should lead to the generation of magnetic fields in the same way we discussed in the case of a first order

EWPT. Unfortunately, to estimate the strength of the magnetic field produced by this mechanism requires the knowledge of the string density and net helicity which, so far, are rather unknown quantities.

4.4 Magnetic helicity and electroweak baryogenesis

As we discussed in the introduction of our report, the cosmological magnetic flux is a nearly conserved quantity due to the high conductivity of the Universe. In this section we will focus on another quantity which, for the same reason, is approximatively conserved during most of the Universe evolution. This is the so called *magnetic helicity* defined by

$$\mathcal{H} \equiv \int d^3x \mathbf{A} \cdot \mathbf{B} \tag{4.60}$$

where $\mathbf{A}$ is the electromagnetic vector potential and $\mathbf{B}$ is the magnetic field. In the presence of a small value of the electric conductivity σ the time evolution of $\mathcal{H}$ is given by

$$\frac{d\mathcal{H}}{dt} = -\frac{1}{\sigma} \int d^3x \; \mathbf{B} \cdot \nabla \times \mathbf{B} \; . \tag{4.61}$$

Besides for the fact to be a nearly conserved quantity, the magnetic helicity is a very interesting quantity for a number of different reasons. The main among these reasons are:

- In a field theory language $\mathcal{H}$ coincides with the Chern-Simon number which is know to be related to the topological properties of the gauge fields.

- Since $\mathcal{H}$ is a P (parity) and CP-odd function, the observation of a nonvanishing net value of this quantity would be a manifestation of a macroscopic violation of both these symmetries.

- It is know from magneto-hydro-dynamics that the presence of magnetic helicity can lead to the amplification of magnetic fields and contribute to their self-organization into a large scale ordered configuration (see Sec.1.4). The same phenomenon could take place at a cosmological level.

In the last few years, several authors proposed mechanisms for the production of magnetic helicity in the early Universe starting from particle physics processes. Cornwall [174] suggested that magnetic helicity was initially stored in the Universe under the form of baryons (B) and leptons (L) numbers possibly generated by some GUT scale baryogenesis mechanism. He assumed that an order one fraction of the total classically conserved $B+L$ charge was dissipated by anomalous processes at the EW scale and showed that a small fraction of this dissipated charge, of the order of $n_{B+L}T^{-3}$, may have been converted into a magnetic helicity of the order of

$$\mathcal{H} \sim \alpha^{-1}(N_B + N_L) \simeq 10^{66} \ \ \text{erg cm} \ . \tag{4.62}$$

Another possibility is that before symmetry breaking of a non-Abelian gauge symmetry vacuum configurations existed which carried nonvanishing winding number. It was shown by Jackiw and Pi [175] that after symmetry breaking, one direction in isospin space is identified with electromagnetism, and the projection of the vacuum configuration becomes a magnetic field with non vanishing helicity.

A different mechanism was proposed by Joyce and Shaposhnikov [176]. In this case it was assumed that some excess of right-handed electrons over left-handed positrons was produced by some means (*e.g.* from some GUT scale leptogenesis) above a temperature T_R. At temperatures higher than T_R perturbative processes which changes electron chirality are out of thermal equilibrium ($T_R \sim 3$ TeV in the SM [177]). Therefore, a chemical potential for right electrons μ_R can be introduced above T_R. On the other hand, the corresponding charge is not conserved because of the Abelian anomaly, which gives

$$\partial_\mu j^\mu_R = -\frac{g'^2 y_R^2}{64\pi^2} \ f_{\mu\nu}\tilde{f}^{\mu\nu} \ . \tag{4.63}$$

In the above, $f_{\mu\nu}$ and $\tilde{f}^{\mu\nu}$ are, respectively, the $U(1)_Y$ hypercharge field strength and its dual, g' is the associated gauge coupling constant, and $y_R = -2$ is the hypercharge of the right electron. As it is well known, Eq.(4.63) relates the variation in the number of the right handed electrons N_R to the variation of the topological properties (Chern-Simon number) of the hypercharge field configuration. By rewriting this expression in terms of the hypermagnetic $\vec{B}_Y$ and of the hyperelectric $\vec{E}_Y$ fields,

$$\partial_\mu j^\mu_R = -\frac{g'^2}{4\pi^2} \ \mathbf{B}_Y \cdot \mathbf{E}_Y \ , \tag{4.64}$$

it is evident the relation of j^μ_R with the hypermagnetic helicity. It is worthwhile to observe that only the hypermagnetic helicity is coupled to the fermion number by the chiral anomaly whereas such a coupling is absent for the Maxwell magnetic helicity because of the vector-like coupling of the electromagnetic field to fermions. From Eq.(4.63) it follows that the variation in N_R, is related to the variation in the Chern-Simon number,

$$N_{CS} = -\frac{g'^2}{32\pi^2}\int d^3\mathbf{x}\epsilon_{ijk}f_{ij}b_k, \tag{4.65}$$

by $\Delta N_R = \frac{1}{2}y_R^2\Delta N_{CS}$. In the above b_k represents the hypercharge field potential. The energy density sitting in right electrons is of order $\mu_R^2T^2$ and their number density of order μ_RT^2. Such fermionic number can be reprocessed into hypermagnetic helicity of order g'^2kb^2, with energy of order k^2b^2, where k is the momentum of the classical hypercharge field and b is its amplitude. Therefore, at $b > T/g'^2$ it is energetically convenient for the system to produce hypermagnetic helicity by "eating-up" fermions. It was showed by Joyce and Shaposhnikov [176], and in more detail by Giovannini and Shaposhnikov [178], that such a phenomenon corresponds to a *magnetic dynamo instability.* In fact, by adding the anomaly term to the Maxwell like equations for the hyperelectric and hypermagnetic fields these authors were able to write the anomalous magneto-hydrodynamical (AMHD) equations for the electroweak plasma in the expanding Universe, including the following generalized hypermagnetic diffusivity equation

$$\frac{\partial\mathbf{B}_Y}{\partial\tau} = -\frac{4a\alpha'}{\pi\sigma}\ \nabla\times(\mu_R\mathbf{B}_Y) +\ \nabla\times(\mathbf{v}\times\mathbf{B}_Y) + \frac{1}{\sigma}\nabla^2\mathbf{B}_Y\ . \tag{4.66}$$

In the above a is the Universe scale factor, $\alpha' \equiv g'^2/4\pi$, $\tau = \int a^{-1}(t)dt$ is the conformal time, and σ the electric conductivity of the electroweak plasma [49]. By comparing Eq.(4.66) with the usual magnetic diffusivity equation (see e.g. Ref.[15]) one sees that the first term on the r.h.s. of Eq.(4.66) corresponds to the so-called dynamo term which is know to be related to the vorticity in the plasma flux [11]. We see that the fermion asymmetry, by providing a macroscopic parity violation, plays a similar role of that played by the fluid vorticity in the dynamo amplification of magnetic fields. In the scenario advocated in Ref.[176, 178], however, it is not clear what are the seed fields from which the dynamo amplification start from. Clearly, no amplification takes place if the initial value of the hypermagnetic field vanishes. Perhaps, seeds field may have been provided by thermal fluctuations or from a previous phase transition although this is a matter of speculation. According to Joyce and Shaposhnikov [176], assuming that a large right electron asymmetry $\mu_R/T \sim 10^{-2}$ was present when $T = T_R$, magnetic field of strength $B \sim 10^{22}$ G may have survived until the EWPT time with typical inhomogeneity scales $\sim 10^6/T$.

Another interesting point raised by Giovannini and Shaposhnikov [178] is that the Abelian anomaly may also process a preexisting hypermagnetic helicity into fermions. In this sense the presence of tangled magnetic fields in the early Universe may provide a new leptogenesis scenario.

Indeed, assuming that a primordial hypermagnetic field $\vec{B}_Y$ was present before the EWPT with some non-trivial topology (i.e. $\langle \mathbf{B}_Y \cdot \nabla \times \mathbf{B}_Y \rangle \neq 0$) the kinetic equation of right electrons for $T > T_c$ is

$$\frac{\partial}{\partial t}\left(\frac{\mu_R}{T}\right) = -\frac{g'^2}{4\pi^2 \sigma a T^3}\frac{783}{88}\mathbf{B}_Y \cdot \nabla \times \mathbf{B}_Y - (\Gamma + \Gamma_{np})\frac{\mu_R}{T}, \tag{4.67}$$

where

$$\Gamma_{np} = \frac{783}{22}\frac{\alpha'^2}{\sigma a \pi^2}\frac{|\mathbf{B}_Y|^2}{T^2} \tag{4.68}$$

is the rate of the N_R nonconserving anomalous processes whereas Γ is the rate of the perturbative ones. In the case $\Gamma_{np} > \Gamma$, as a consequence of Eq.(4.67), one finds

$$n_R \simeq -\frac{88\pi^2}{783 g'^2}\left(\frac{\mathbf{B}_Y \cdot \nabla \times \mathbf{B}_Y}{|\mathbf{B}_Y|^2}\right) + O\left(\frac{\Gamma}{\Gamma_{np}}\right) \ . \tag{4.69}$$

Below the critical temperature the hypermagnetic fields are converted into ordinary Maxwell magnetic fields . Similarly to the usual EW baryogenesis scenario, the fermion number asymmetry produced by the Abelian anomaly may survive the sphaleron wash-out only if the EWPT is strongly first order, which we know to be incompatible with the standard model in the absence of primordial magnetic fields. However, Giovannini and Shaposhinikov argued that this argument might not apply in the presence of strong magnetic fields (we shall discuss this issue in Chap.5). If this is the case a baryon asymmetry compatible with the observations might have been generated at the EW scale. Another prediction of this scenario is the production of strong density fluctuations at the BBN time which may affect the primordial synthesis of light elements [179].

Primordial magnetic fields and the primordial magnetic helicity may also have been produced by the interaction of the hypercharge component of the electromagnetic field with a cosmic pseudoscalar field condensate which provides the required macroscopic parity violation. This idea was first sketched by Turner and Widrow [45] in the framework of an inflationary model of the Universe which we shall discuss in more details in the next section. Turner and Widrow assumed the pseudoscalar field to be the axion, a particle which existence is invoked for the solution of the strong CP problem (for a review see Ref. [180]). Although the axion is supposed to be electrically neutral it couples to electromagnetic field by means of the anomaly. Indeed, the effective Lagrangian for axion electrodynamics is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\theta\partial^\mu\theta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g_a\theta F_{\mu\nu}\tilde{F}^{\mu\nu} \ , \tag{4.70}$$

where g_a is a coupling constant of order α, the vacuum angle $\theta = \phi_a/f_a$, ϕ_a is the axion field, f_a is the Peccei-Quinn symmetry breaking scale (see Ref.[180]), $F_{\mu\nu}$ is the electromagnetic field strength and $\tilde{F}^{\mu\nu}$ is its dual. Since the axion field, as any other scalar field, is not conformally invariant (see the next section), it will be amplified during the inflationary expansion of the Universe starting from quantum fluctuations, giving rise to $\langle\theta^2\rangle \sim (H_0/f_a)^2$, which can act as a source term for the electromagnetic field [5].

Carroll and Field [182] reconsider in more details the idea of Turner and Widrow and found that the evolution of a Fourier mode of the magnetic field with wave number k is governed by the equation

$$\frac{d^2F_\pm}{d\tau^2} + \left(k^2 \pm g_a k\frac{d\phi_a}{d\tau}\right)F_\pm = 0 \tag{4.71}$$

where $F_\pm = a^2(B_x \pm iB_y)$ are the Fourier modes corresponding to different circular polarizations, and τ is the conformal time. One or both polarization modes will be unstable for $k < g_a|d\phi_a/d\tau|$, whereas both polarization modes can becomes unstable to exponential growth if ϕ is oscillating. In this case it seems as if a quite strong magnetic field could be produced during

[5] The careful reader may wonder what is the fate of axions in the presence of cosmic magnetic fields. Interestingly, it was showed by Ahonen, Enqvist and Raffelt [181] that although oscillating cosmic axions drives an oscillating electric field, the ensuing dissipation of axions is found to be inversely proportional to the plasma conductivity and is, therefore, negligible.

inflation. However, such a conclusion was recently criticized by Giovannini [183] who noted that above the EWPT temperature QCD sphalerons [184] are in thermal equilibrium which can effectively damp axion oscillations. In fact, because of the presence of QCD sphaleron the axion equation of motion becomes

$$\ddot{\phi}_a + (3H + \gamma)\,\dot{\phi}_a = 0 \tag{4.72}$$

where [184]

$$\gamma = \frac{\Gamma_{sphal}}{f_a^2 T} \simeq \frac{\alpha_s^4 T^3}{f_a^2} \tag{4.73}$$

(where $\alpha_s = g_s^2/4\pi$). Giovannini founds that sphaleron induced damping dominates over damping produced by the expansion of the Universe if $f_a > 10^9$ GeV. Since astrophysical and cosmological bounds [125] leave open a window 10^{10} GeV $< f_a < 10^{12}$ GeV, it follows that no magnetic fields amplification was possible until QCD sphaleron went out of thermal equilibrium. A very tiny magnetic helicity production from axion oscillations may occur at lower temperatures. In fact, Giovannini [183] estimated that in the temperature range 1 GeV $> T >$ 10MeV a magnetic helicity of the order of

$$\frac{\langle \mathbf{B} \cdot \nabla \times \mathbf{B} \rangle}{\sigma \rho_c} \sim 10^{-22} \tag{4.74}$$

may be generated, which is probably too small to have any phenomenological relevance.

Generation of magnetic fields from coherent oscillations or roll-down of pseudoscalar fields different from the axion has been also considered in the literature. It is interesting that pseudoscalar fields with an axion-like anomalous coupling to the electromagnetic field appear in several possible extension of the SM. Typically these fields have only perturbative derivative interactions and therefore vanishing potential at high temperatures, and acquire a potential at lower temperatures through non-perturbative interactions. The generic form of this potential is $V(\phi) = V_0^4\ A(\phi/f)$, where A is a bounded function of the "Peccei-Quinn" scale f which in this case can be as large as the Planck scale. The pseudoscalar mass $m \sim V_0^2/f$ may range from a few eV to 10^{12} GeV. The amplification of magnetic fields proceeds in a way quite similar to that discussed for axions. The evolution equation of the electromagnetic Fourier modes were derived by Brustein and Oaknin [185] who added to Eq.(4.71) the effect of a finite electric conductivity, finding

$$\frac{d^2 F_\pm}{d\tau^2} + \sigma \frac{\partial F_\pm}{d\tau} + \left(k^2 \pm gk \frac{d\phi}{d\tau} \right) F_\pm = 0 \ . \qquad (4.75)$$

If the pseudoscalar field is oscillating, the field velocity $d\phi/d\tau$ changes sign periodically and both polarization modes are amplified, each during a different semi-cycle. Each mode is amplified during one part of the cycle and damped during the other part of the cycle. Net amplification results when $gkd\phi/d\tau > \sigma^2$ and the total amplification is exponential in the number of cycles. Taking into account that $\sigma \sim 10T$ before the EWPT [49], Brustein and Oaknin found that a huge amplification can be obtained from pseudoscalar field oscillations at $T \sim 1$ TeV, for a scalar mass m of few TeV and $gf \sim 10$. This is a particularly interesting range of parameters as in this case the pseudoscalar field dynamics could be associated with breaking of supersymmetry at the TeV scale. Since only a limited range of Fourier modes are amplified, with k not too different from T, and modes with $k/T < \tau_{EW}\sigma \ll 1$ are rapidly dissipated, amplified fields may survive until the EWPT only if amplification occurred just before the transition. If this is a natural assumption may be a matter of discussion. However, it was pointed-out by Brustein and Oaknin [185] that if, depending on the form of the pseudoscalar field potential, this field rolls instead of oscillating, then the hypermagnetic fields would survive until EWPT even if the amplification takes place at higher temperatures before the transition. In this case there could be interesting consequences for the EW baryogenesis. This is so, because the pseudoscalar field may carry a considerable amount of helicity. According to what discussed in the above, this number will be released in the form of fermions and baryons if the EWPT is strongly first order. Brustein and Oaknin argued that this mechanism could naturally generate the observed BAU.

Clearly, a more serious problem of this mechanism is the same of other EW baryogenesis scenarios, namely to have a strongly first order EWPT. An interesting possibility which was proposed by two different groups [178, 186] is that strong magnetic fields may enhance the strength of the EWPT (see Chap.5). Unfortunately, detailed lattice computations [187] showed that this is not the case. Furthermore, in a recent work by Comelli et al. [188] it was shown that strong magnetic fields also increase the rate of EW sphalerons so that the preservation of the baryon asymmetry calls for a much stronger phase transition than required in the absence of a magnetic field. The authors of Ref.[188] showed that this effect overwhelms the gain in the phase transition strength (see Sec.5.3.2). Therefore, the only way for the kind of EW baryogenesis mechanism discussed in the above to work is to invoke for extensions of the standard model which allow for a strong first order EWPT [144].

Electroweak strings, which we introduced in the previous section, may also carry a net hypermagnetic helicity and act as a source of the observed BAU. One of the interesting features of this objects is that they should have been formed during the EWPT even if this transition is second order or just a cross over [173]. It is known that electroweak strings can carry magnetic helicity, hence a baryon number, which is related to the twisting and linking of the string gauge fields [159, 189]. Several authors (see *e.g.* Ref.[190]) tried to construct a viable EW baryogenesis scenario based on these embedded defects. Many of these models, however, run into the same problems of more conventional EW baryogenesis scenarios. An interesting attempt was done by Barriola [191] who invoked the presence of a primordial magnetic field at the EWPT time. Barriola observed that the production and the following decay of electroweak strings give rise, in the presence of an external magnetic field, to a variation in the baryon number ΔB in the time interval dt, given by

$$\Delta B = \frac{N_f}{32\pi^2}\int dt \int d^3x \; \alpha^2 \cos 2\theta_W \vec{E}_Z \cdot \vec{B}_Z + \frac{\alpha^2}{2}\sin^2\theta_W \left(\vec{E}\cdot\vec{B}_Z + \vec{E}_Z\cdot\vec{B}\right) \; , \tag{4.76}$$

where $\vec{E}_Z$ and $\vec{B}_Z$ are the Z-electric and Z-magnetic fields of the strings whereas $\vec{E}$ and $\vec{B}$ are the corresponding standard Maxwell fields. The first term in the r.h.s. of (4.76) represents the change in the helicity of the string network, and the second and third terms come form the coupling of the string fields with the external Maxwell fields. Whereas, the first term averages to zero over a large number of strings, the other terms may not. Clearly, some bias is required to select a direction in the baryon number violation. In the specific model considered by Barriola this is obtained by a CP violation coming from the extension of the Higgs sector of the SM, and from the dynamics of strings which are supposed to collapse along their axis. It was concluded by Barriola that such a mechanism could account for the BAU. Unfortunately, it is quite unclear if the invoked out-of-equilibrium mechanism based on the string collapse could indeed be effective enough to avoid the sphaleron wash-out. We think, however, that this possibility deserves further study.

4.5 Magnetic fields from inflation

As noted by Turner and Widrow [45] inflation (for a comprehensive introduction to inflation see Ref.[46]) provides four important ingredients for the production of primeval magnetic fields .

- Inflation naturally produces effects on very large scales, larger than the Hubble horizon, starting from microphysical processes operating on a causally connected volume. If electromagnetic quantum fluctuations are amplified during inflation they could appear today as large-scale static magnetic fields (electric field should be screened by the high conductivity plasma).

- Inflation also provides the dynamical means to amplify these long-wavelength waves. If the conformal invariance of the electromagnetic field is broken in some way (see below) magnetic fields could be excited during the de Sitter Universe expansion. This phenomenon is analogous to particle production occurring in a rapidly changing space-time metric.

- During inflation (and perhaps during most of reheating) the Universe is not a good conductor so that magnetic flux is not conserved and the ratio r of the magnetic field with the radiation energy densities can increase.

- Classical fluctuations with wavelength $\lambda \lesssim H^{-1}$ of massless, minimally coupled fields can grow *super-adiabatically*, i.e. their energy density decrease only as $\sim a^{-2}$ rather than a^{-4}.

The main obstacle on the way of this nice scenario is given by the fact that in a conformally flat metric, like the Robertson-Walker usually considered, the background gravitational field does not produce particles if the underlying theory is conformally invariant [192]. This is the case for photons since the classical electrodynamics is conformally invariant in the limit of vanishing fermion masses. Several ways out this obstacle have been proposed. Turner and Widrow [45] considered three possibilities. The first is to break explicitly conformal invariance by introducing a gravitational coupling, like $RA_\mu A^\mu$ or

$R_{\mu\nu}A^{\mu}A^{\mu}$, where R is the curvature scalar, $R_{\mu\nu}$ is the Ricci tensor, and A^{μ} is the electromagnetic field. These terms breaks gauge invariance and give the photons an effective, time-dependent mass. In fact, one of the most severe constraints to this scenario come from the experimental upper limit to the photon mass, which today is $m_{\gamma} < 2 \times 10^{-16}$ eV [193]. Turner and Widrow showed that for some suitable (though theoretically unmotivated) choice of the parameters, such a mechanism may give rise to galactic magnetic fields even without invoking the galactic dynamo. We leave to the reader to judge if such a booty deserve the abandonment of the theoretical prejudice in favor of gauge invariance. A different model invoking a spontaneous breaking of gauge symmetry of electromagnetism, implying nonconservation of the electric charge, in the early stage of the evolution of the Universe has been proposed by Dolgov and Silk [194].

The breaking of the conformal invariance may also be produced by terms of the form $R_{\mu\nu\lambda\kappa}F^{\mu\nu}F^{\lambda\kappa}/m^2$ or $RF^{\mu\nu}F_{\mu\nu}$, where m is some mass scale required by dimensional considerations. Such terms arise due to one-loop vacuum polarization effects in curved space-time, and they have the virtue of being gauge invariant. Unfortunately, Turner and Widrow showed that they may account only for a far too small contribution to primordial magnetic fields. The third way to break conformal invariance discussed by Turner and Widrow invoke a coupling of the photon to a charged field which is not conformally coupled or the anomalous coupling to a pseudoscalar. This mechanism was already illustrated in the previous section.

The anomaly can give rise to breaking of the conformal invariance also in a different way. The kind of anomaly we are now discussing about is the conformal anomaly, which is related to the triangle diagram connecting two photons to a graviton. It is known (for a review see Ref.[195]) that this kind of diagrams breaks conformal invariance by producing a nonvanishing trace of the energy-momentum tensor

$$T^{\mu}_{\mu} = \frac{\alpha\beta}{8\pi} F^{a}_{\mu\nu} F^{a\ \mu\nu} \ , \tag{4.77}$$

where α is the fine-structure constant of the theory based on the $SU(N)$ gauge-symmetry with N_f fermion families, and

$$\beta = \frac{11}{3}N - \frac{2}{3}N_f \ . \tag{4.78}$$

Dolgov [196] pointed-out that such an effect may lead to strong electromagnetic fields amplification during inflation. In fact, Maxwell equations are modified by the anomaly in the following way

$$\partial_\mu F^\mu_\nu + \beta \frac{\partial_\mu}{a} F^\mu_\nu = 0 \tag{4.79}$$

which, in the Fourier space, gives rise to the equation

$$A'' + \beta \frac{a'}{a} A' + k^2 A = 0 \ , \tag{4.80}$$

where A is the amplitude of the vector potential, and a prime stands for a derivation respect to the conformal time τ. At the inflationary stage, when $a'/a = -1/\tau$ Dolgov found a solution of (4.80) growing like $\left(\frac{H}{k}\right)^{\beta/2}$. Since k^{-1} grows well above the Hubble radius during the de Sitter phase, a huge amplification can be obtained if $\beta > 0$. Dolgov showed that for $\beta \sim 1$ the magnetic field generated during the inflationary stage can be large enough to give rise to the observed fields in galaxies even without a dynamo amplification. Unfortunately, such a large value of β seems to be unrealistic ($\beta \approx 0.06$ for SU(5) with three charged fermions). The conclusion is that galactic magnetic fields might be produced by this mechanism only invoking a group larger than $SU(5)$ with a large number of fermion families, and certainly no without the help of dynamo amplification.

As we discussed in the above conformal invariance of the electromagnetic field is generally spoiled whenever the electromagnetic field is coupled to a scalar field. Ratra [197] suggested that a coupling of the form $e^{\kappa\phi}F^{\mu\nu}F_{\mu\nu}$, where κ is a arbitrary parameter, may lead to a huge amplification of electromagnetic quantum fluctuations into large scale magnetic fields during inflation. Such a coupling is produced in some peculiar models of inflation with an exponential inflaton potential [198]. It should be noted by the reader that the scalar field ϕ coincide here with the inflaton field. According to Ratra, present time intergalactic magnetic fields as large as 10^{-9} G may be produced by this mechanism which would not require any dynamo amplification to account for the observed galactic fields. Unfortunately, depending on the parameter of the underling model, the predicted field could also be as low as 10^{-65} G !

A slightly more predictive, and perhaps theoretically better motivated, model has been proposed independently by Lemoine and Lemoine [199] , and Gasperini, Giovannini and Veneziano [200], which is based on superstring cosmology [201]. This model is based on the consideration that in string theory the electromagnetic fields is coupled not only to the metric ($g_{\mu\nu}$), but also to the dilaton field ϕ. In the low energy limit of the theory, and after dimensional reduction from 10 to 4 space-time dimensions, such a coupling takes the form

$$\sqrt{-g}e^{-\phi}F^{\mu\nu}F_{\mu\nu} \tag{4.81}$$

which breaks conformal invariance of the electromagnetic field and coincides with the coupling considered by Ratra [197] if $\kappa = -1$. Ratra, however, assumed inflation to be driven by the scalar field potential, which is not the case in string cosmology. In fact, typical dilaton potentials are much too steep to produce the required slow-roll of the inflaton (= dilaton) field. According to string cosmologists, this problem can be solved by assuming inflation to be driven by the kinetic part of the dilaton field, i.e. from ϕ' [201]. In such a scenario the Universe evolves from a flat, cold, and weakly coupled ($\phi = -\infty$) initial unstable vacuum state toward a curved, dilaton-driven, strong coupling regime. During this period, called *pre-big-bang* phase, the scale factor and of the dilaton evolve as

$$a(\tau) \sim (-\tau)^{\beta}, \quad \phi(\tau) \sim \kappa \ln a, \quad \tau < -\tau_1 \ , \tag{4.82}$$

with $\beta > 1$ and $\kappa < 0$. At $\tau > -\tau_1$ it begins the standard FRW phase with a radiation dominated Universe. In the presence of the non-trivial dilaton background the modified Maxwell equations takes the form [199]

$$\nabla_\mu F^{\mu\nu} - \nabla_\mu \phi \ F^{\mu\nu} = 0 \ . \tag{4.83}$$

Electromagnetic field amplification from quantum fluctuations take place during the *pre-big-bang* phase when $\phi' = \delta/\tau$, where $\delta = \beta\kappa$. By following the evolution of the electromagnetic field modes from $t = -\infty$ to now, Lemoine and Lemoine estimated that, in the most simple model of dilaton-driven inflation (with $V(\phi) = \rho = p = 0$) a very tiny magnetic field is predicted today

$$\langle B^2 \rangle^{1/2} \sim 10^{-62} \left(\frac{L}{1\mathrm{Mpc}}\right)^{-2.1} \left(\frac{H_1}{M_{Pl}}\right)^{0.07} \ \mathrm{G} \ , \tag{4.84}$$

where $H_1 = 1/\tau_1$, which is far too small to account for galactic fields.

Gasperini, Giovannini and Veneziano [200], reached a different conclusion, claiming that magnetic fields as large those required to explain galactic fields without dynamo amplification may be produced on the protogalactic scale.

The reason for such a different result is that they assumed a new phase to exist between the dilaton-dominated phase and the FRW phase during which dilaton potential is nonvanishing. The new phase, called the *string phase*, should start when the string length scale λ_S becomes comparable to the horizon size at the conformal time τ_S [202]. Unfortunately, the duration of such a phase is quite unknown, which makes the model not very predictive.

Recently, several papers have been published (see e.g. Refs.[203, 204, 205]) which proposed the generation of magnetic fields by fluctuations of scalar (or pseudo-scalar) fields which were amplified during, or at the end, of inflation. Typically, all these models predict magnetic fields which need dynamo amplification to account for the observed galactic and cluster fields.

4.6 Magnetic fields from cosmic strings

At the beginning of this chapter we shortly discussed the Harrison-Rees mechanism for vortical production of magnetic fields in the primordial plasma. In this section we will shortly review as cosmic string may implement this mechanism by providing a vorticity source on scales comparable to galactic sizes. Cosmic strings are one-dimensional topological defects which are supposed to have been formed during some primordial phase transition through the Kibble mechanism [153], which we already discussed in Sec.4.3 (for a review on cosmic strings see Ref. [206]). The idea that cosmic string may produce plasma vorticity and magnetic fields was first proposed by Vachaspati and Vilenkin [207]. In this scenario vorticity is generated in the wakes of fast moving cosmic strings after structure formation begins. Differently from Harrison's [131] scenario (see Sec.4.1), in this case vorticity does decay with Universe expansion since the vortical eddies are gravitationally bounded to the string. Even if the mechanism we are considering is supposed to take place after recombination, a sufficient amount of ionization should be produced by the violent turbulent motion so that the Harrison-Rees [19] mechanism can still operate. The scale of coherence of the generated magnetic fields is set by the scale of wiggles of the string and, for wakes created at recombination time it can be up to 100 kpc. The predicted field strength is of the order of $\sim 10^{-18}$ G, that could enough to seed the galactic dynamo. The main problem with this scenario is that is not clear whether stable vortical motion can be really generated by the chaotic motion of the string wiggles.

An alternative mechanism has been proposed by Avelino and Shellard

[208]. In their model, vorticity is generated not by the wiggles but by the strings themselves which, because of the finite dynamical friction, drag matter behind them inducing circular motions over inter-string scales. Unfortunately, the magnetic field strength predicted by this model at the present time is very weak $\sim 10^{-23}$ G, which can only marginally seed the galactic dynamo.

Larger fields may be produced if cosmic strings are superconducting. String superconductivity was first conceived by Witten [209]. The charge carriers on these strings can be either fermions or bosons, and the critical currents can be as large as 10^{20} Ampere. If primordial magnetic fields preexist, or formed together with, cosmic string they may play a role charging up superconducting string loops and delaying their collapse [210]. Otherwise, superconducting cosmic strings can themselves give rise to magnetic fields in a way similar to that proposed by Avelino and Shellard. An important difference, however, arises with respect to the non-superconducting case discussed in Ref.[208]. As shown by Dimopoulos and Davis [211], superconducting strings networks may be more tangled and slower than conventional cosmic strings because of the strong current which increases dynamical friction. It was shown by Dimopoulos [212] that if the string velocity is small enough the gravitational influence of the string on the surrounding plasma becomes a relevant effect. As a consequence, plasma is dragged by the string acquiring substantial momentum. Such a momentum may induce turbulence which could generate magnetic fields on scale of the order of the inter-string distance. Such a distance is smaller than the conventional cosmic string distance. Quite strong magnetic fields may be produced by this mechanism. The only known constraint come from the requirement that the string network does not produce too large temperature anisotropies in the CMBR. By imposing this constraint, Dimopoulos estimated that present time magnetic fields as large as 10^{-19} G with a coherence scale of ~ 1 Mpc may be produced.

Contrarily to previous claims, in a very recent paper by Voloshin it was showed that the generation of large scale magnetic fields by domain walls is not possible [215].

Chapter 5

Particles and their couplings in the presence of strong magnetic fields

In the previous chapter we have seen that, very strong magnetic fields could have been produced in the early Universe. Here we investigate the effects of such strong fields on bound states of quarks and on condensates created by spontaneous symmetry breaking. For example, we have already seen in Chap.3 that strong magnetic fields can affect masses and decay rates of charged particles and modify the rate of weak processes. As we already discussed in Sec.1.4 another crucial issue concerns the stability of strong magnetic fields. QED allows the existence of arbitrary large magnetic fields provided matter constituents have spin less than $\frac{1}{2}$ [38].

This is so because the Lorentz force cannot perform any work on charged particles so that real particle-antiparticles free pairs cannot be produced. In Sec.3.1 we have already seen that quantum corrections do not spoil this classical argument. We have also seen that although pair production can be catalyzed by strong magnetic fields at finite temperature and density, in this case it is the heat-bath that pays for the energy cost of the effect. This situation changes when, in the presence of very strong fields, QCD and electroweak corrections cannot be disregarded. We shall see that the QCD and electroweak field allow for the formation of condensates of charged particles with no energy cost. This may lead to screening of magnetic, or hypermagnetic, fields resembling the Meissner effect in superconductors.

5.1 Low Lying States for Particles in Uniform Magnetic Fields

Following Refs.[216, 217] in this section we will consider questions related to the mass shifts and decay of bound states of quarks. Mass shifts occur both due to the effect that magnetic fields have on the strong binding forces, and due to the direct interactions of charged spinning particles with external fields. The modifications of the strong forces are such as to close the gap between the proton and neutron masses and ultimately make the proton heavier. A delicate interplay between the anomalous magnetic moments of the proton and neutron drives the mass shifts due to the direct interactions in the same direction. For $B > 1.5 \times 10^{18}$ G the neutron becomes stable and as the field is increased past 2.7×10^{18} G the proton becomes unstable to decay into a neutron, positron and neutrino.

The quantum mechanics of a Dirac particle with no anomalous magnetic moment in a uniform external magnetic field is straightforward. We shall present the results for the case where particles do have such anomalous moments. In reality, in fields so strong that the mass shifts induced by such fields are of the order of the mass itself one cannot define a magnetic moment as the energies are no longer linear in the external field. Schwinger [37] calculated the self energy of an electron in an external field and we shall use here his results. We cannot follow this procedure for the proton or neutron as we do not have a good field theory calculation of the magnetic moments of these particles, even for small magnetic fields; all we have at hand is a phenomenological anomalous magnetic moment. However, for fields that change the energies of these particles by only a few percent, we will consider these as point particles with the given anomalous moments. In the Sec.5.1.7 we will discuss possible limitations of this approach.

5.1.1 Protons in an External Field

The Dirac Hamiltonian for a proton with a uniform external magnetic field $\mathbf{B}$ is

$$H = \boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}(\mathbf{r})) + \beta M_p - \frac{e}{2M_p}\left(\frac{g_p}{2} - 1\right)\beta \boldsymbol{\Sigma} \cdot \mathbf{B}\,. \tag{5.1}$$

The vector potential $\mathbf{A}(\mathbf{r})$ is related to the magnetic field by $\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{r} \times \mathbf{B}$ and $g_p = 5.58$ is the proton's Landé g factor. We first solve this equation for the case where the momentum along the magnetic field direction is zero and then boost along that direction till we obtain the desired momentum. For $\mathbf{B}$ along the $\mathbf{z}$ direction and $p_z = 0$ the energy levels are [38]

$$E_{n,m,s} = \left[2eB(n + \frac{1}{2}) - eBs + {M_p}^2\right]^{\frac{1}{2}} - \frac{e}{2M_p}\left(\frac{g_p}{2} - 1\right) Bs\,. \tag{5.2}$$

In the above, n denotes the Landau level, m the orbital angular momentum about the magnetic field direction and $s = \pm 1$ indicates whether the spin is along or opposed to that direction; the levels are degenerate in m. $n = 0$ and $s = +1$ yield the lowest energy

$$E = \tilde{M}_p = M_p - \frac{e}{2M_p}\left(\frac{g_p}{2} - 1\right) B\,. \tag{5.3}$$

As we shall be interested in these states only we will drop the n and s quantum numbers. The Dirac wave function for this state is

$$\psi_{m,p_z=0}(\mathbf{r}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \phi_m(x, y)\,, \tag{5.4}$$

ϕ_m's are the standard wave functions of the lowest Landau level;

$$\phi_m(x, y) = \frac{\left[\frac{1}{2}|eB|\right]^{\frac{m+1}{2}}}{\sqrt{\pi m!}}[x + iy]^m \exp\left[-\frac{1}{4}|eB|(x^2 + y^2)\right]\,. \tag{5.5}$$

Boosting to a finite value of p_z is straightforward; we obtain

$$E_m(p_z) = \sqrt{{p_z}^2 + \tilde{M}^2}\,, \tag{5.6}$$

with a wave function

$$\psi_{m,p_z}(\mathbf{r}) = \begin{pmatrix} \cosh\theta \\ 0 \\ \sinh\theta \\ 0 \end{pmatrix} \frac{e^{ip_z z}}{\sqrt{2\pi}} \phi_m(x, y)\,, \tag{5.7}$$

where 2θ, the rapidity, is obtained from $\tanh 2\theta = p_z/E_m(p_z)$.

In the non-relativistic limit the energy becomes

$$E_m(p_z) = \tilde{M} + \frac{{p_z}^2}{2\tilde{M}} \tag{5.8}$$

and the wave function reduces to

$$\psi_{m,p_z}(\mathbf{r}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{e^{ip_z z}}{\sqrt{2\pi}} \phi_m(x,y)\,. \tag{5.9}$$

5.1.2 Neutrons in an External Field

For a neutron the Dirac Hamiltonian is somewhat simpler

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta M_n - \frac{e}{2M_n}\left(\frac{g_n}{2}\right)\beta \boldsymbol{\Sigma} \cdot \mathbf{B}\,. \tag{5.10}$$

with $g_n = -3.82$. Again for $p_z = 0$ the states of lowest energy, the ones we shall be interested in, have energies

$$E(\mathbf{p}_\perp, p_z = 0) = \frac{e}{2M_n}\left(\frac{g_n}{2}\right) B + \sqrt{{\mathbf{p}_\perp}^2 + {M_n}^2} \tag{5.11}$$

Boosting to a finite p_z we obtain

$$E(\mathbf{p}) = \sqrt{E(\mathbf{p}_\perp, p_z = 0)^2 + {p_z}^2}\,. \tag{5.12}$$

The wave functions corresponding to this energy are

$$\psi_{\mathbf{p}}(\mathbf{r}) = \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{r}}}{(2\pi)^{\frac{3}{2}}} u(\mathbf{p}, s = -1)\,, \tag{5.13}$$

where $u(\mathbf{p}, s = -1)$ is the standard spinor for a particle with momentum $\mathbf{p}$, energy $\sqrt{\mathbf{p}^2 + {M_n}^2}$ (not the energy of Eq. (5.12)) and spin down.

In the non-relativistic limit

$$E(\mathbf{p}) = M_n + \frac{e}{2M_n}\left(\frac{g_n}{2}\right) B + \frac{\mathbf{p}^2}{2M_n} \tag{5.14}$$

and the wave functions are

$$\psi_{\mathbf{p}}(\mathbf{r}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{r}}}{(2\pi)^{\frac{3}{2}}}\,. \tag{5.15}$$

5.1.3 Electrons in an External Field

We might be tempted to use, for the electron, the formalism used for the proton with the Landé factor replaced by $g_e = 2 + \alpha/\pi$. However as we shall see for magnetic fields sufficiently strong as to make the proton heavier than the neutron the change in energy of the electron would appear to be larger than the mass of the electron itself. The point particle formalism breaks down and we have solve QED, to one loop, in a strong magnetic field; fortunately this problem was treated by Schwinger[37]. The energy of an electron with $p_z = 0$, spin up and in the lowest Landau level is

$$E_{m,p_z=0} = M_e \left[1 + \frac{\alpha}{2\pi} \ln\left(\frac{2eB}{{M_e}^2}\right)\right] . \tag{5.16}$$

For field strengths of subsequent interest this correction is negligible; the energy of an electron in the lowest Landau level, with spin down and a momentum of p_z is

$$E_{m,p_z} = \sqrt{{p_z}^2 + {M_e}^2} \, . \tag{5.17}$$

and with wave function similar to those of the proton

$$\psi_{m,p_z}(\mathbf{r}) = \begin{pmatrix} 0 \\ \cosh\theta \\ 0 \\ \sinh\theta \end{pmatrix} \frac{e^{ip_z z}}{\sqrt{2\pi}} \phi^*_m(x,y) \, , \tag{5.18}$$

where the boost rapidity, 2θ, is defined below Eq. (5.7) while the Landau level wave function is defined in Eq. (5.5). The reason the complex conjugate wave function appears is that the electron charge is opposite to that of the proton.

5.1.4 Effects of Magnetic Fields on Strong Forces

We must be sure that shifts due to changes of color strong forces will not shift states in the opposite direction. The best method to study masses of QCD bound states is the use of sum rules [218]. This method uses the SVZ [219] generalized short distance expansion that includes not only perturbative pieces, but also higher dimensional operators like the chiral and gluon condensates reflecting the non Abelian nature of the vacuum. Fortunately the proton has a simple structure [218] which reflects the fact that if chiral symmetry is restored the proton and neutron masses vanish.

$$M_{p,n} = 3a \langle q\bar{q}\rangle^{\frac{1}{3}} + \text{small corrections} \tag{5.19}$$

where a is a constant. Meson mass terms are more involved; for example the ρ mass is

$$M_\rho = b\,(\text{perturbative terms}) + c\,\langle G_{\mu\nu}G^{\mu\nu}\rangle + d\,\langle q\bar{q}\rangle\,. \tag{5.20}$$

b, c and d are constants of comparable magnitude [218]. As we shall show it is only the change of $\langle q\bar{q}\rangle$ due to external magnetic fields that may be obtained in a reliable manner.

In the presence of external fields we expect the chiral condensates for quarks of different charges to vary and Eq. (5.19) becomes

$$\begin{aligned} M_p^3 &= a\left(2\langle u\bar{u}\rangle + \langle d\bar{d}\rangle\right)\,, \\ M_n^3 &= a\left(2\langle d\bar{d}\rangle + \langle u\bar{u}\rangle\right)\,. \end{aligned} \tag{5.21}$$

To first order in condensate changes we find

$$\begin{aligned} \delta M_p &= \frac{M_p}{9}\left(2\frac{\delta\langle u\bar{u}\rangle}{\langle u\bar{u}\rangle} + \frac{\delta\langle d\bar{d}\rangle}{\langle d\bar{d}\rangle}\right)\,, \\ \delta M_n &= \frac{M_n}{9}\left(2\frac{\delta\langle d\bar{d}\rangle}{\langle d\bar{d}\rangle} + \frac{\delta\langle u\bar{u}\rangle}{\langle u\bar{u}\rangle}\right)\,. \end{aligned} \tag{5.22}$$

Combining

$$\delta M_p - \delta M_n = \frac{M}{9}\left(\frac{\delta\langle u\bar{u}\rangle}{\langle u\bar{u}\rangle} - \frac{\delta\langle d\bar{d}\rangle}{\langle d\bar{d}\rangle}\right)\,. \tag{5.23}$$

A simple method for studying the behavior of the chiral condensates in the presence of external constant fields is through the use of the Nambu-Jona-Lasinio model [220]. This has been done by Klevansky and Lemmer [221] and a fit to their results is

$$\langle q\bar{q}\rangle(B) = \langle q\bar{q}\rangle(0)\left[1 + \left(\frac{e_q B}{\Lambda^2}\right)^2\right]^{\frac{1}{2}}\,, \tag{5.24}$$

with $\Lambda = 270$ MeV and e_q the charge on the quark. To lowest order we find

$$\frac{\delta\langle q\bar{q}\rangle}{\langle q\bar{q}\rangle} = \frac{1}{2}\left(\frac{e_q B}{\Lambda^2}\right)^2\,. \tag{5.25}$$

and

$$\delta M_p - \delta M_n = \frac{M}{54}\left(\frac{eB}{\Lambda^2}\right)^2 \; ; \tag{5.26}$$

As in the previous section, these corrections are such as drive the proton energy up faster than that of the neutron. One can understand the sign of this effect; the radius of a quark-antiquark pair will decrease with increasing magnetic field thus making the condensate larger. As the u quark has twice the charge of the d quark, its condensate will grow faster and as there are more u quarks in the proton than in the neutron its mass will increase faster.

The fact that our estimate of the sign of the neutron-proton mass difference is the same as that due to electromagnetic effects is crucial. QCD sum rules and our method of evaluating the chiral condensates are both crude and the magnitude of the mass difference is uncertain. Had the sign of the hadronic correction been opposite, cancellations could have occurred and the argument for a narrowing of the mass difference and ultimate reversal could not have been made. We are quite sure of the sign of the chiral change. The magnetic field acts in the naive way in the spin of the scalar bound state in the condensate so it is sure that goes as calculated [222].

5.1.5 Proton Life Time

First we study the decay kinematics.

Combining Eq. (5.3), Eq. (5.14) and Eq. (5.26) we find the proton-neutron energy difference as a function of the applied magnetic field,

$$\Delta(B) = \left(-1.3 + 0.38 B_{14} + 0.11 B_{14}^2\right) \text{ MeV}\,. \tag{5.27}$$

B_{14} is the strength of the magnetic field in units of 10^{14} T (1 T = 10^4 G). The neutron becomes stable for $B > 1.5 \times 10^{14}$ T and the proton becomes unstable to β decay for $B > 2.7 \times 10^{14}$ T. We shall now turn to a calculation of the life time of the proton for fields satisfying the last inequality.

With the wave functions of the various particles in the magnetic fields we may define field operators for these particles. For the proton and electron we shall restrict the summation over states to the lowest Landau levels with spin up, down respectively; for magnetic fields of interest the other states will not contribute to the calculation of decay properties. For the same reason, the neutron field will be restricted to spin down only. The proton and neutron kinematics will be taken as non-relativistic.

$$\Psi_p(\mathbf{r}) = \sum_m \int dp_z \left[a_m(p_z) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{e^{ip_z z}}{\sqrt{2\pi}} \phi_m(x,y) \right.$$

$$\left. + b^\dagger{}_m(p_z) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \frac{e^{-ip_z z}}{\sqrt{2\pi}} \phi_m(x,y) \right] , \tag{5.28}$$

with $\phi_m(x,y)$ defined in Eq. (5.5) and the energy, $E_m(p_z)$ in Eq. (5.8). $a_m(p_z)$ is the annihilation operator for a proton with momentum $p_z\mathbf{z}$ and angular momentum m; $b_m(p_z)$ is the same for the negative energy states. For the neutron the field is

$$\Psi_n(\mathbf{r}) = \int d^3p \left[a(\mathbf{p}) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{r}}}{(2\pi)^{\frac{3}{2}}} + b^\dagger(\mathbf{p}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}}{(2\pi)^{\frac{3}{2}}} \right] , \tag{5.29}$$

with an obvious definition of the annihilation operators. For the electron we use fully relativistic kinematics and the field is

$$\Psi_e(\mathbf{r}) = \sum_m \int dp_z \sqrt{\frac{M_e}{E}} \left[a_m(p_z) \begin{pmatrix} 0 \\ \cosh\theta \\ 0 \\ \sinh\theta \end{pmatrix} \frac{e^{ip_z z}}{\sqrt{2\pi}} \phi^*_m(x,y) \right.$$

$$\left. + b^\dagger_m(p_z) \begin{pmatrix} 0 \\ \cosh\theta \\ 0 \\ \sinh\theta \end{pmatrix} \frac{e^{-ip_z z}}{\sqrt{2\pi}} \phi^*_m(x,y) \right] . \tag{5.30}$$

5.1.6 Decay Rates and Spectrum

The part of the weak Hamiltonian responsible for the decay $p \to n + e^+ + \nu_e$ is

$$H = \frac{G_F}{\sqrt{2}} \int d^3x \bar{\Psi}_n \gamma_\mu (1+\gamma_5) \Psi_p \bar{\Psi}_\nu \gamma^\mu (1+\gamma_5) \Psi_e \, . \tag{5.31}$$

For non-relativistic heavy particles the matrix element of this Hamiltonian between a proton with quantum numbers $p_z = 0$, $m = m_i$, a neutron with momentum $\mathbf{p}_n$, a neutrino with momentum $\mathbf{p}_\nu$ and an electron in state $m = m_f$ and with $p_{z,e}$ is

$$\langle H \rangle = \frac{2G_F}{(2\pi)^3} \left(\frac{E_e + p_{z,e}}{E_e - p_{z,e}} \right)^{\frac{1}{4}} \sin(\theta_\nu/2) \sqrt{\frac{M_e}{E_e}} \delta(p_{z,e} + p_{z,\nu} + p_{z,n})$$
$$\int dx\, dy\, \phi^*_{m_f}(x,y) \phi_{m_i}(x,y) \exp[-i(\boldsymbol{p}_{\perp,n} + \boldsymbol{p}_{\perp,\nu}) \cdot \boldsymbol{r}_\perp] \, ; \qquad (5.32)$$

θ_ν is the azimuthal angle of the neutrino. The integral in the above expression can be evaluated in a multipole expansion. Note that the natural extent of the integral in the transverse direction is $1/\sqrt{eB}$ whereas the neutron momenta are, from Eq. (5.27), of the order of $\sqrt{eB(0.12 + 0.04B_{14})}$; thus setting the exponential term in this integral equal to one will yield a good estimate for the rate and spectrum of this decay. The positron spectrum is given by

$$\frac{d\Gamma}{dp_{z,e}} = \frac{4}{3} \frac{G_F^2 M_p}{(2\pi)^6} \frac{E_e + p_{z,e}}{E_e} (\Delta - E_e)^3 \, ; \qquad (5.33)$$

where Δ is defined in Eq. (5.27). For $\Delta \gg M_e$ the total rate is easily obtained

$$\Gamma = \frac{2}{3} \frac{G_F^2 M_p}{(2\pi)^6} \Delta^4 \, . \qquad (5.34)$$

For $B = 5 \times 10^{14}$ T, the lifetime is $\tau = 6$ s.

5.1.7 Caveats and Limitations

For general magnetic fields we expect the masses of particles to be nonlinear functions of these fields. Such an expression has been obtained, to order α, for the electron[37]. For small fields this reduces to a power series, up to logarithmic terms, in B/B_c, where B_c is some scale. For the electron $B_c = m_e^2/e$. For the hadronic case the value of B_0 is uncertain. $B_c = M_p^2/e = 1.7 \times 10^{16}$ T is probably too large and M_p should be replaced by a quark constituent mass and e be e_q; in that case $B_c = (2-4) \times 10^{15}$ T, depending on the quark type. This is also the range of values of Λ^2/e_q in Eq. (5.24). The effects we have studied need fields around a few $\times 10^{14}$ T or an order of magnitude smaller than the lowest candidate for B_c. Eq. (5.27) bay be viewed as a power series expansion up to terms of order $(B/B_c)^2$; as the coefficient of the quadratic term was obtained from a fit to a numerical solution, logarithms of B/B_c may be hidden in the coefficient. As in Ref. [37], even powers will be spin independent and the odd ones will be linear in the spin direction and may be viewed as field dependent corrections to the magnetic moment. We cannot prove, but only hope, that the coefficient of the $(B/B_c)^3$ term, the first correction to the magnetic moment is not unusually large; should it turn out to be big and of opposite sign to the linear and quadratic terms, the conclusions of this analysis would be invalidated. These arguments, probably, apply best to the field dependence of the magnetic moments of the quarks rather than the total moment of the baryons. We may ask what is the effect on these magnetic moments due to changes in the "orbital" part of the quark wave functions. To first order we expect no effect as all the quarks are in S states and there is no orbital contribution to the total moment. The next order perturbation correction will be *down* by $(r_b/r_c)^4$ compared to the leading effect; r_b is a hadronic radius and r_c is the quarks cyclotron radius. This again contributes to the $(B/B_c)^3$ term in the expansion for the energy of a baryon.

Another limitation is due to the results of Ref. [217] where it is shown that fields of the order of a few $\times 10^{14}$ T are screened by a changes in chiral condensates. In fact, as the chiral condensate will, in large fields, point in the charged π direction, the baryonic states will not have a definite charge. Whether the proton-neutron reversal takes place for fields below those that are screened by chiral condensates or vice versa is a subtle question; the approximations used in this paper and in Ref. [217] are not reliable to give an unambiguous answer. The treatment of the effects of magnetic fields on the strong force contributions to the baryon masses relies on the Nambu-Jona-Lasinio model and in Ref.[217] the variation of f_π with magnetic field was not taken into account. It is clear from this discussion and the one from the previous paragraph that we cannot push the results of this calculation past few $\times 10^{14}$ T.

Experimental Consequences

The mass evolution of protons, neutrons and electrons in magnetic fields and due to electromagnetism alone, will force a proton to decay in a very intense field. Including the effects of chiral condensates diminishes the field even further. Qualitatively, it is clear that the effect enhances the electromagnetic contribution but its exact value depends on the model. This points to a novel astrophysical mechanism for creation of extra galactic positrons.

5.1.8 Proton neutron mass difference by a lattice calculation

One can take another approach, in principle exact, to calculate the mass difference using lattice gauge theory. We introduce the magnetic field in the lattice by multiplying a link $U_\mu(x)$ with a phase $U^B_\mu(x)$. Fixing the field,for convenience, in the direction z we then set:

$$U^B_x(x) = exp(-ieBa^2 yL_x) \text{ for } x = L_x - 1; = \text{ otherwise} \tag{5.35}$$

$$U^B_y(x) = \exp(ieBa^2(x - x_0) \tag{5.36}$$

where x_0 is an offset for the magnetic field.

Consequently the plaquette in the x-y plane is

$$P^B(x) = \exp(ieBa^2(-L_x L_y + 1)) \text{ for } x = L_x - 1, y = L_y - 1 \tag{5.37}$$

$$P^B(x) = exp(ieBa^2) \text{ otherwise} \tag{5.38}$$

The magnetic field is homogeneous only if B is is quantized as $a^2 eB = 2\pi n(L_x L_y)^{-1}$ This is a troublesome condition since the field is very large for reasonable lattice size. In the simulations we ignored this condition inducing some inhomogeneity. The results are very preliminary: at this stage it is difficult to find these effects with present lattice technology, but in principle it is possible. For details of the calculations see [224]. This method is still not efficient given the present state of the art in simulations.

5.2 Screening of Very Intense Magnetic Fields by chiral symmetry breaking

Now we discuss another interesting phenomenon if very strong fields could be created.

In very intense magnetic fields, $B > 1.5 \times 10^{18}$ G, the breaking of the strong interaction $SU(2) \times SU(2)$ symmetry arranges itself so that instead of the neutral σ field acquiring a vacuum expectation value it is the charged π field that does and the magnetic field is screened.

In the previous section we discussed that fields with complicated interactions of non-electromagnetic origin can induce various instabilities in the presence of very intense magnetic fields. By very intense we mean 10^{18} G to 10^{24} G. Fields with anomalous magnetic moments [223] or fields coupled by transition moments [216] may induce vacuum instabilities. The usual breaking of the strong interactions, chiral symmetry (χSM), is incompatible with very intense magnetic fields. Using the standard $SU(2)\times SU(2)$ chiral σ model we show that magnetic fields $B \geq B_c$ with $B_c = \sqrt{2} m_\pi f_\pi$ are screened; $f_\pi = 132$ MeV is the pion decay constant and m_π is the mass of the charged pions. This result is opposite to what occurs in a superconductor; in that case it is weak fields that are screened and large ones penetrate and destroy the superconducting state.

As the magnetic fields are going to be screened we must be very careful in how we specify an external field. One way would be to give f_π a spatial dependence and take it to vanish outside some large region of space. In the region that f_π vanishes we could specify the external field and see how it behaves in that part of space where chiral symmetry is broken. This is the procedure used in studying the behavior of fields inside superconductors. In the present situation we find this division artificial and, instead of specifying the magnetic fields, we shall specify the external currents. Specifically we will look, at first, at the electromagnetic field coupled to the charged part of the σ model and to the current I in a long straight wire. From this result it will be easy to deduce the behavior in other current configurations. We will discuss a solenoidal current configuration towards the end of this work.

The Hamiltonian density for this problem is

$$\begin{aligned} H = \quad & \frac{1}{2}\boldsymbol{\nabla}\sigma \cdot \boldsymbol{\nabla}\sigma + \frac{1}{2}\boldsymbol{\nabla}\pi_0 \cdot \boldsymbol{\nabla}\pi_0 + (\boldsymbol{\nabla} + e\mathbf{A})\pi^\dagger \cdot (\boldsymbol{\nabla} - e\mathbf{A})\pi \\ + \quad & g(\sigma^2 + \pi\cdot\pi - f_\pi^2)^2 + m_\pi^2(f_\pi - \sigma) + \frac{1}{2}(\boldsymbol{\nabla}\times \boldsymbol{A})^2 - \mathbf{j}\cdot\mathbf{A}\,; \quad (5.39) \end{aligned}$$

$\mathbf{j}$ is the external current. We have used cylindrical coordinates with $\boldsymbol{\rho}$ the two dimensional vector normal to the z direction. We will study this problem in the limit of very large g, where the radial degree of freedom of the chiral field is frozen out and we may write

$$\begin{aligned} \sigma &= f_\pi \cos\chi\,, \\ \pi_0 &= f_\pi \sin\chi\cos\theta\,, \\ \pi_x &= f_\pi \sin\chi\sin\theta\cos\phi\,, \end{aligned}$$

$$\pi_y \quad = f_\pi \sin\chi \sin\theta \sin\phi \,. \tag{5.40}$$

In terms of these variables the Hamiltonian density becomes

$$\begin{aligned} H = \quad & \frac{f_\pi^2}{2}(\boldsymbol{\nabla}\chi)^2 + \frac{f_\pi^2}{2}\sin^2\chi(\boldsymbol{\nabla}\theta)^2 + \frac{f_\pi^2}{2}\sin^2\chi\sin^2\theta(\boldsymbol{\nabla}\phi - e\mathbf{A})^2 \\ + \quad & m_\pi^2 f_\pi^2(1-\cos\chi) + \frac{1}{2}(\boldsymbol{\nabla}\times \boldsymbol{A})^2 - \mathbf{j}\cdot\mathbf{A}\,. \end{aligned} \tag{5.41}$$

The angular field ϕ can be eliminated by a gauge transformation. For a current along a long wire we have

$$\mathbf{j} = I\delta(\boldsymbol{\rho})\mathbf{z}\,; \tag{5.42}$$

The vector potential will point along the z direction, $\mathbf{A} = A\mathbf{z}$ and the fields will depend on the radial coordinate only. The equations of motion become

$$\begin{aligned} -\boldsymbol{\nabla}^2\chi + \sin\chi\cos\chi(\boldsymbol{\nabla}\theta)^2 + e^2\sin\chi\cos\chi\sin^2\theta A^2 + m_\pi^2\sin\chi = & \quad 0\,, \\ \boldsymbol{\nabla}(\sin^2\chi\boldsymbol{\nabla}\theta) + e^2\sin^2\chi\sin\theta\cos\theta A^2 = & \quad 0\,, \\ -\boldsymbol{\nabla}^2 A + e^2 f_\pi^2\sin^2\chi\sin^2\theta A - I\delta(\boldsymbol{\rho}) = & \quad 0 \end{aligned} \tag{5.43}$$

In the absence of the chiral field the last of the Eqs. (5.43) gives the classical vector potential due to a long wire

$$A = \frac{I}{2\pi}\ln\frac{\rho}{a}\,, \tag{5.44}$$

with a an ultraviolet cutoff. The energy per unit length in the z direction associated with this configuration is

$$E = \frac{I^2}{4\pi}\ln\frac{R}{a}\,, \tag{5.45}$$

where R is the transverse extent of space (an infrared cutoff).

Before discussing the solutions of (5.43) it is instructive to look at the case where there is no explicit chiral symmetry breaking, $m_\pi = 0$. The solution that eliminates the infrared divergence in Eq. (5.45) is $\chi = \theta = \pi/2$ and A satisfying

$$-\boldsymbol{\nabla}^2 A + e^2 f_\pi^2 A - I\delta(\boldsymbol{\rho}) = 0\,. \tag{5.46}$$

For any current the field A is damped for distances $\rho > 1/ef_\pi$ and there is no infrared divergence in the energy. (Aside from the fact that chiral symmetry is broken explicitly, the reason the above discussion is only of pedagogical value is that the coupling of the pions to the quantized electromagnetic field does break the $SU(2) \times SU(2)$ symmetry into $SU(2) \times U(1)$ and the charged pions get a light mass, $m_\pi \sim 35$ MeV [193], even in the otherwise chiral symmetry limit.)

The term in Eq. (5.41) responsible for the pion mass prevents us from setting $\chi = \pi/2$ everywhere; the energy density would behave as $\pi f_\pi^2 m_\pi^2 R^2$, an infrared divergence worse than that due to the wire with no chiral field present. We expect that χ will vary from $\pi/2$ to 0 as ρ increases and that asymptotically we will recover classical electrodynamics. Although we cannot obtain a closed solution to Eqs. (5.43), if the transition between $\chi = \pi/2$ and $\chi = 0$ occurs at large ρ, we can find an approximate solution. The approximation consists of neglecting the $(\boldsymbol{\nabla}\chi)^2$ term in Eq. (5.41); we shall return to this shortly. The solution of these approximate equations of motion is

$$\begin{aligned} \chi &= \begin{cases} \frac{\pi}{2} & \text{for } \rho < \rho_0 \\ 0 & \text{for } \rho > \rho_0 \,, \end{cases} \\ \theta &= \frac{\pi}{2} \,, \\ A &= \begin{cases} -\frac{I}{2\pi}\left[K_0(ef_\pi\rho) - I_0(ef_\pi\rho)K_0(ef_\pi\rho_0)/I_0(ef_\pi\rho_0)\right] & \text{for } \rho < \rho_0 \\ \frac{I}{2\pi}\ln\frac{\rho}{\rho_0} & \text{for } \rho > \rho_0 \,; \end{cases} \end{aligned} \tag{5.47}$$

ρ_0 is a parameter to be determined by minimizing the energy density of Eq. (5.41). Note that for $\rho > \rho_0$ the vector potential as well as the field return to values these would have in the absence of any chiral fields and that for $\rho < \rho_0$ the magnetic field decreases exponentially as $B \sim \exp(-ef_\pi\rho)$. The physical picture is that, as in a superconductor, near $\rho = 0$ a cylindrical current sheet is set up that opposes the current in the wire and there is a return current near $\rho = \rho_0$; Ampère's law insures that the field at large distances is as discussed above. The energy density for the above configuration, neglecting the spatial variation of χ, is

$$H = -\frac{I^2}{4\pi}\left[\frac{K_0(ef_\pi\rho_0)}{I_0(ef_\pi\rho_0)} + \ln(ef_\pi\rho_0)\right] + \pi m_\pi^2 f_\pi^2 \rho^2 + \cdots \,, \tag{5.48}$$

where the dots represent infrared and ultraviolet regulated terms which are, however, independent of ρ_0. For $\rho_0 > 1/ef_\pi$ the term involving the Bessel functions may be neglected and minimizing the rest with respect to ρ_0 yields

$$\rho_0 = \frac{I}{2\sqrt{2}\pi m_\pi f_\pi}\,. \tag{5.49}$$

This is the main result of this work.

We still have to discuss the validity of the two approximations we have made. The neglect of the Bessel functions in Eq. (5.48) is valid for $ef_\pi\rho_0 > 1$ which in turn provides a condition on the current I, $eI/m_\pi > 2\sqrt{2}\pi$ or more generally

$$I/m_\pi >> 1\,. \tag{5.50}$$

The same condition permits us to neglect the spatial variation of χ around $\rho = \rho_0$. Let χ vary from $\pi/2$ to 0 in the region $\rho - d/2$ to $\rho + d/2$, with $1/d$ of the order of f_π or m_π. The contribution of the variation of χ to the energy density is $\Delta H = \pi^3 f_\pi^2 \rho_0 d$. Eq. (5.50) insures that ΔH is smaller than the other terms in Eq. (5.48).

Eq.(5.49) has a very straightforward explanation. It results from a competition of the magnetic energy density $\frac{1}{2}B^2$ and the energy density of the pion mass term $m_\pi^2 f_\pi^2(1-\cos\chi)$. The magnetic field due to the current I is $B = I/2\pi\rho$ and the transition occurs at $B = B_c$, with $B_c = \sqrt{2}m_\pi f_\pi$. The reader may worry that the magnetic fields very close to such thin wires are so large as to invalidate completely the use of the chiral model as a low energy effective QCD theory. In order to avoid this problem we may consider the field due to a solenoid of radius R. The field is zero outside the solenoid, $\mathbf{B} = B(\rho)\mathbf{z}$ inside with $B(R) = B_0$. At no point does the field become unboundedly large. Using the same approximations as previously we obtain the following solutions of the equations of motion (for $B_0 \geq B_c$)

$$\begin{aligned}
\chi &= \begin{cases} 0 & \text{for } \rho > R \\ \frac{\pi}{2} & \text{for } R > \rho > \rho_0 \\ 0 & \text{for } \rho < \rho_0\,, \end{cases} \\
\theta &= \frac{\pi}{2}\,, \\
B &= \begin{cases} 0 & \text{for } \rho > R \\ a_1 K_0(ef_\pi\rho) + a_2 I_0(ef_\pi\rho) & \text{for } R > \rho > \rho_0 \\ B_c & \text{for } \rho < \rho_0\,; \end{cases}
\end{aligned} \tag{5.51}$$

Continuity of the vector potential determines the coefficients a_1 and a_2,

$$\begin{aligned}
a_1 &= \left[B_0 I_1(ef_\pi\rho_0) - B_c I_1(ef_\pi R)\right]/D(R,\rho_0)\,, \\
a_2 &= \left[B_0 K_1(ef_\pi R) - B_c K_1(ef_\pi\rho_0)\right]/D(R,\rho_0)\,;
\end{aligned} \tag{5.52}$$

$D(R, \rho_0) = K_1(ef_\pi R)I_1(ef_\pi \rho_0) - K_1(ef_\pi \rho_0)I_1(ef_\pi R)$ and ρ_0 is determined, once more, by minimizing the energy. For $(R,\ \rho_0) > 1/ef_\pi$

$$\rho_0 = R - \frac{1}{ef_\pi} \ln (B_0/B_c) \ . \tag{5.53}$$

For $B_0 \leq B_c$, $\chi = 0$ everywhere and $B(\rho) = B_0$ in the interior of the solenoid. Thus, for any current configuration, the chiral fields will adjust themselves to screen out fields larger than B_c. Topological excitations may occur in the form of magnetic vortices; the angular field ϕ of Eq. (5.40) will wind around a quantized flux tube of radius $1/ef_\pi$ [171].

5.3 The effect of strong magnetic fields on the electroweak vacuum

It was pointed out by Ambjorn and Olesen [223] (see also Ref.[225]) that the Weinberg-Salam model of electroweak interactions shows an instability at $B \simeq 10^{24}$ Gauss. The nature of such instability can be understood by looking at the expression of the energy of a particle with electric charge e, and spin $\mathbf{s}$, moving in homogeneous magnetic field $\mathbf{B}$ directed along the z-axis. As we already discussed in Sec.3.1, above a critical field $B_c = m^2/e$ particle energy is discretized into Landau levels

$$E_n^2 = k_z^2 + (2n+1)e|\mathbf{B}| - 2e\mathbf{B}\cdot\mathbf{s} + m^2 \ . \tag{5.54}$$

We observe that energy of scalar ($s = 0$) and spinor ($s_z = \pm 1/2$) is always positive, and indeed no instability arise in QED (it is possible to verify that quantum one-loop corrections do not spoil this conclusion). In the case of vector particles ($s_z = 0, \pm 1$), however, the lowest energy level ($n = 0,\ k_z = 0,\ s_z = +1$) becomes imaginary for $B > B_c$, which could be the signal of vacuum instability. The persistence of imaginary values of the one loop corrected lowest level energy [223] seems to confirm the physical reality of the instability.

As it is well known the Weinberg-Salam model contains some charged vector fields, namely the $W^\pm$ gauge bosons. The coupling of the W_μ field to an external electromagnetic field A_μ^{ext} is given by

$$\mathcal{L}_{int} = -\frac{1}{4}|F_{\mu\nu}^{ext}|^2 - -\frac{1}{2}|D_\mu W_\nu - D_\nu W_\mu|^2 - m_W^2 W_\mu^\dagger W^\mu - ieF_{\mu\nu}^{ext} W^\mu W^\nu \tag{5.55}$$

with

$$D_\mu = \partial_\mu - ieA_\mu^{ext} \; . \tag{5.56}$$

The important term in the previous expression is the "anomalous" magnetic moment term $ieF_{\mu\nu}^{ext}W^\mu W^\nu$, which arises because the non-Abelian nature of the $SU(2)$ component of the Weinberg-Salam model gauge group structure. Due to this term the mass eigenvalues of the W Lagrangian becomes

$$m^2 = m_W^2 \pm eB \; . \tag{5.57}$$

As expected from the considerations in the above, a tachyonic mode appears for $B > B_c$. The corresponding eigenvector for zero kinetic energy is determined by solving the equation of motions

$$D_i W_j - D_j W_i = 0 \quad i,\; j = x, y \; , \tag{5.58}$$

where $W_{1,2} = W_x \pm iW_y$. Ambjorn and Olesen argued that a suitable solution of this equation is

$$|W(x,y)| = e^{-\frac{1}{4}m_W(x^2+y^2)} \; , \tag{5.59}$$

corresponding to a vortex configuration where W-fields wind around the z-axis. This configuration corresponds to the Nielsen-Olesen vortex solution [157]. A similar phenomenon should also take place for Z bosons. Given the linearity of the equations of motion it is natural to assume that a superpositions of vortices is formed above the critical field. This effect resemble the behaviour of a type-II superconductor in the presence of a critical field magnetic field. In that case $U(1)$ symmetry is locally broken by the formations of a lattice of Abrikosov vortices in the Cooper-pairs condensate through which the magnetic field can flow. In the electroweak case this situation is reversed, with the formation of a W condensate along the vortices. Concerning the back-reaction of the W condensate on the magnetic field, an interesting effect arises. By writing the electric current induced by the W fields

$$j_\mu(W) = 2ie\left(W^\dagger D_\mu W - W D_\mu W^\dagger\right) \; , \tag{5.60}$$

Ambjorn and Olesen noticed that its sign is opposite to the current induced by the Cooper pairs in a type-II superconductor, which is responsible for the Meissner magnetic field screening effect. Therefore, they concluded that the W-condensate induce *anti-screening* of the external magnetic field.

Although the Higgs field Φ does not couple directly to the electromagnetic field (this is different from the case of a superconductor where the Cooper-pairs condensate couples directly to A_μ^{ext}), it does through the action of the W condensate. This can be seen by considering the Higgs, W potential in the presence of the magnetic fields:

$$V(\phi, W) = 2\left(eB - m_W^2\right)|W|^2 + g^2\phi^2|W|^2 - 2\lambda\phi_0^2\phi^2 + 2g^2|W|^4 + \lambda\left(\phi_+^4\phi_0^4\right) . \tag{5.61}$$

In the above ϕ_0 and ϕ_+ are respectively the Higgs field vev and charged component, g is the SU(2) coupling constant, and λ is the Higgs the self-interaction coupling constant. We see that the W-condensate influences the the Higgs field at classical level due to the $\phi^2|W|^2$ term. It is straightforward to verify that if $eB < m_W^2 = \frac{1}{2}g^2\phi_0^2$ the minimum of $V(\phi, W)$ sits in the standard field value $\phi = \phi_0$ with no W condensate. Otherwise a W condensate is energetically favoured with the minimum of the potential sitting in

$$\phi_{min}^2 = \phi_0^2\,\frac{m_H^2 - eB}{m_H^2 - m_W^2} \tag{5.62}$$

where

$$m_H^2 \equiv 4\lambda\phi_0^2, \qquad m_W^2 \equiv \frac{1}{2}g^2\phi_0^2 . \tag{5.63}$$

We see that the Higgs expectation value will vanish as the average magnetic field strength approaches zero, provided the Higgs mass is larger than the W mass. This seems to suggest that a W-condensate should exist for

$$m_W^2 < eB < m_H^2 , \tag{5.64}$$

and that the $SU(2) \times U_Y(1)$ symmetry is restored above $H_c^{(2)} \equiv m_H^2/e$. Thus, anti-screening should produce restoration of the electroweak symmetry in the core of W vortices. If $m_H < m_W$ the electroweak vacuum is expected to behave like a type I superconductor with the formation of homogeneous W-condensate above the critical magnetic field. The previous qualitative conclusion have been confirmed by analytical and numerical computations performed for $m_H = m_W$ in Ref.[223], and for arbitrary Higgs mass in Refs.[226, 227].

A different scenario seems, however, to arise if thermal corrections are taken into account. Indeed, recent finite temperature lattice computations [187] showed no evidence of the Ambjorn and Olesen phase. According to

Skalozub and Demchik [228] such a behaviour may be explained by properly accounting the contribution of Higgs and gauge bosons daisy diagrams to the effective finite temperature potential.

In conclusion, it is quite uncertain if the Ambjorn and Olesen phenomenon was really possible in the early Universe.

5.3.1 The electroweak phase transition in a magnetic field

We shall now consider the possible effects of strong magnetic fields on the electroweak phase transition (EWPT). As it is well known, the properties of the EWPT are determined by the Higgs field effective potential. In the framework of the minimal standard model (MSM), taking into account radiative corrections from all the known particles and for finite temperature effects, one obtains that

$$V_{\rm eff}(\phi, T) \simeq -\frac{1}{2}(\mu^2 - \alpha T^2)\phi^2 - T\delta\phi^3 + \frac{1}{4}(\lambda - \delta\lambda_T)\phi^4 \ . \tag{5.65}$$

where ϕ is the radial component of the Higgs field and T is the temperature (for the definitions of the coefficients see *e.g.* Ref.[186]).

A strong hypermagnetic field can produce corrections to the effective potential as it affects the charge particles propagators (see below). There is, however, a more direct and simpler effect of magnetic and hypermagnetic fields on the EWPT which was recently pointed-out by Giovannini and Shaposhnikov [178] and by Elmfors, Enqvist and Kainulainen [186]. The authors of Refs.[178, 186] noticed that hypermagnetic fields affect the Gibbs free energy (in practice the pressure) difference between the broken and the unbroken phase, hence the strength of the transition. The effect can be understood by the analogy with the Meissner effect, *i.e.* the expulsion of the magnetic field from superconductors as consequence of photon getting an effective mass inside the specimen. In our case, it is the Z–component of the hypercharge $U(1)_Y$ magnetic field which is expelled from the broken phase. This is just because Z–bosons are massive in that phase. Such a process has a cost in terms of free energy. Since in the broken phase the hypercharge field decomposes into

$$A^Y_\mu = \cos\theta_w A_\mu - \sin\theta_w Z_\mu \ , \tag{5.66}$$

we see that the Gibbs free energy in the broken and unbroken phases are

$$G_b = V(\phi) - \frac{1}{2}\cos^2\theta_w (B_Y^{ext})^2 \ , \tag{5.67}$$

$$G_u = V(0) - \frac{1}{2}(B_Y^{ext})^2 \ . \tag{5.68}$$

where B_Y^{ext} is the external hypermagnetic field. In other words, compared to the case in which no magnetic field is present, the energy barrier between unbroken and broken phase, hence the strength of the transition, is enhanced by the quantity $\frac{1}{2}\sin^2\theta_w(B_Y^{ext})^2$. According to the authors of Refs.[178, 186] this effect can have important consequence for baryogenesis.

In any scenario of baryogenesis it is crucial to know at which epoch do the sphaleronic transitions, which violate the sum $(B+L)$ of the baryon and lepton numbers, fall out of thermal equilibrium. Generally this happens at temperatures below $\bar{T}$ such that [229]

$$\frac{E(\bar{T})}{\bar{T}} \geq A \ , \tag{5.69}$$

where $E(T)$ is the sphaleron energy at the temperature T and $A \simeq 35 - 45$, depending on the poorly known prefactor of the sphaleron rate. In the case of baryogenesis at the electroweak scale one requires the sphalerons to drop out of thermal equilibrium soon after the electroweak phase transition. It follows that the requirement $\bar{T} = T_c$, where T_c is the critical temperature, turns eq. (5.69) into a lower bound on the Higgs vacuum expectation value (VEV),

$$\frac{v(T_c)}{T_c} \geq 1 \, . \tag{5.70}$$

As we already discussed, it is by now agreed [143] that the standard model (SM) does not have a phase transition strong enough as to fulfill Eq.(5.70), whereas there is still some room left in the parameter space of the minimal supersymmetric standard model (MSSM) [144].

The interesting observation made in Refs.[178, 186] is that a magnetic field for the hypercharge $U(1)_Y$ present for $T > T_c$ may help to fulfill Eq.(5.70). In fact, it follows from the Eqs.(5.67), that in presence of the magnetic field the critical temperature is defined by the expression

$$V(0, T_c) - V(\phi, T_c) = \frac{1}{2}\sin^2\theta_w (B_Y^{ext}(T_c))^2 \ . \tag{5.71}$$

This expression implies a smaller value of T_c than that it would take in the absence of the magnetic field, hence a larger value of the ratio (5.70).

Two major problems, however, bar the way of this intriguing scenario. The first problem is that by affecting fermion, Higgs and gauge field propagators, the hypermagnetic field changes the electroweak effective potential in a nontrivial way. Two different approaches have been used to estimate the relevance of this kind of effects based either on lattice simulations [143] or analytical computations [228]. Both approaches agreed in the conclusion that for a Higgs field mass compatible with the experimental constraints ($m_H > 75$ GeV), and for field strengths B, $B_Y \lesssim 10^{23}$ G, the standard model EWPT is second order or a cross-over. Although this negative result could, perhaps, be overcome by adopting a supersymmetrical extension of the standard model (see e.g. Ref.[144]), a second, and more serious problem arises by considering the effect of the magnetic field on the anomalous processes (sphalerons) which are responsible for lepton and baryon violation at the weak scale. This effect will be the subject of the next section.

5.3.2 Sphalerons in strong magnetic fields

The sphaleron, is a static and unstable solution of the field equations of the electroweak model, corresponding to the top of the energy barrier between two topologically distinct vacua [230]. In the limit of vanishing Weinberg angle, $\theta_w \to 0$, the sphaleron is a spherically symmetric, hedgehog-like configuration of $SU(2)$ gauge and Higgs fields. No direct coupling of the sphaleron to a magnetic field is present in this case. As θ_w is turned on, the $U_Y(1)$ field is excited and the spherical symmetry is reduced to an axial symmetry. A very good approximation to the exact solution is obtained using the Ansatz by Klinkhamer and Laterveer [231], which requires four scalar functions of r only,

$$\begin{aligned}
&g' a_i\, dx^i = (1 - f_0(\xi))\, F_3\,, \\
&g W_i^a \sigma^a\, dx^i = (1 - f(\xi))(F_1\sigma^1 + F_2\sigma^2) + (1 - f_3(\xi))F_3\sigma^3\,, \\
&\mathbf{\Phi} = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ h(\xi) \end{pmatrix}\,,
\end{aligned} \tag{5.72}$$

where g and g' are the $SU(2)_L$ and $U(1)_Y$ gauge couplings, v is the Higgs VEV such that $M_W = gv/2$, $M_h = \sqrt{2\lambda}v$, $\xi = gvr$, σ^a $(a = 1, 2, 3)$ are the Pauli matrices, and the F_a's are 1-forms defined in Ref. [230]. The boundary conditions for the four scalar functions are

$$\begin{aligned} f(\xi)\,,\, f_3(\xi)\,,\, h(\xi) \to 0 \quad f_0(\xi) \to 1 \qquad & for \;\; \xi \to 0 \\ f(\xi)\,,\, f_3(\xi)\,,\, h(\xi)\,,\, f_0(\xi) \to 1 \qquad & for \;\; \xi \to \infty\,. \end{aligned} \tag{5.73}$$

It is known [230, 231] that for $\theta_w \neq 0$ the sphaleron has some interesting electromagnetic properties. In fact, differently from the pure $SU(2)$ case, in the physical case a nonvanishing hypercharge current J_i comes-in. At the first order in θ_w, J_i takes the form

$$J_i^{(1)} = -\frac{1}{2} g' v^2 \frac{h^2(\xi)[1 - f(\xi)]}{r^2} \epsilon_{3ij} x_j \ , \tag{5.74}$$

where h and f are the solutions in the $\theta_w \to 0$ limit, giving for the dipole moment

$$\mu^{(1)} = \frac{2\pi}{3} \frac{g'}{g^3 v} \int_0^\infty d\xi \xi^2 h^2(\xi)[1 - f(\xi)]\ . \tag{5.75}$$

The reader should note that the dipole moment is a true electromagnetic one because in the broken phase only the electromagnetic component of the hypercharge field survives at long distances.

Comelli et al. [188] considered what happens to the sphaleron when an external hypercharge magnetic field, B_Y^{ext}, is turned on. They found that the energy functional is modified as

$$E = E_0 - E_{dip}\,, \tag{5.76}$$

with

$$E_0 = \int d^3x \left[\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{4} f_{ij} f_{ij} + (D_i \mathbf{\Phi})^\dagger (D_i \mathbf{\Phi}) + V(\mathbf{\Phi})\right] \tag{5.77}$$

and

$$E_{\rm dip} = \int d^3x J_i A_i^Y = \frac{1}{2} \int d^3x f_{ij} f_{ij}^c \tag{5.78}$$

with $f_{ij} \equiv \partial_i A_j^Y - \partial_j A_i^Y$. A constant external hypermagnetic field B_Y^{ext} directed along the x_3 axis was assumed. In the $\theta_w \to 0$ limit the sphaleron has no hypercharge contribution and then $E_{\rm dip}^{(0)} = 0$. At $O(\theta_w)$, using (5.74) and (5.75) the authors of Ref.[188] got a simple magnetic dipole interaction energy

$$E_{\rm dip}^{(1)} = \mu^{(1)} B_Y^{ext}\ . \tag{5.79}$$

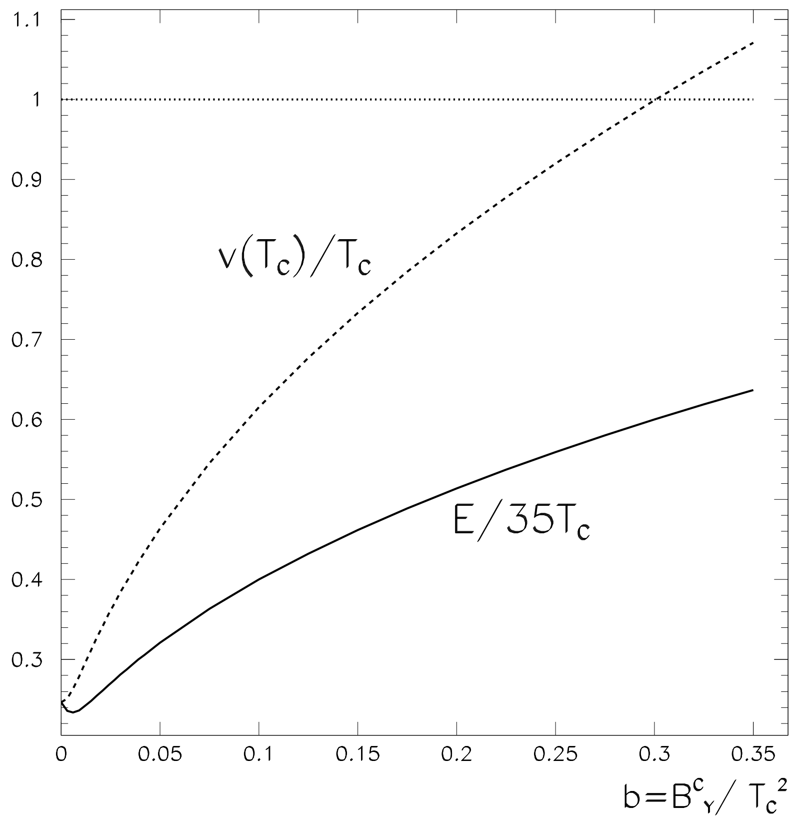

Figure 5.1: The VEV at the critical temperature, $v(T_c)$, and the sphaleron energy *vs.* the external magnetic field for $M_h = M_W$. We see that even if $v(T_c)/T_c \gtrsim 1$ the washout condition $E/T_c \gtrsim 35$ is far from being fulfilled. From Ref.[188]

In order to assess the range of validity of the approximation (5.79) one needs to go beyond the leading order in θ_w and look for a nonlinear B_Y^{ext}–dependence of E. This requires to solve the full set of equations of motion for the gauge fields and the Higgs in the presence of the external magnetic field. Fortunately, a uniform B_Y^{ext} does not spoil the axial symmetry of the problem. Furthermore, the equation of motion are left unchanged ($\partial_i f_{ij}^{ext} = 0$) with respect to the free field case. The only modification induced by B_Y^{ext} resides in the boundary conditions since – as $\xi \to \infty$ – we now have

$$f(\xi)\,, h(\xi) \to 1\,, \qquad f_3(\xi)\,, f_0(\xi) \to 1 - B_Y^{ext} \sin 2\theta_w \frac{\xi^2}{8gv^2} \tag{5.80}$$

whereas the boundary condition for $\xi \to 0$ are left unchanged.

The solution of the sphaleron equation of motions with the boundary conditions in the above were determined numerically by the authors of Ref.[188]. They showed that in the considered B_Y^{ext}–range the corrections to the linear approximation

$$\Delta E \simeq \mu^{(1)} \cos\theta_w B_Y^{ext}$$

are less than 5%. For larger values of B_Y^{ext} non-linear effects increase sharply. However, as we discussed in the previous sections, for such large magnetic fields the broken phase of the SM is believed to become unstable to the formation either of W-condensates [223] or of a mixed phase [187]. In such situations the sphaleron solution does not exist any more. Therefore, it is safe to limit the previous analysis to values $B_Y^{ext} \leq 0.4\, T^2$.

The reduction of sphaleron energy due to the interaction with the field B_Y^c has relevant consequences on the sphaleronic transition rate which is increased with respect to free field case. As a consequence, in an external magnetic field the relation between the Higgs VEV and the sphaleron energy is altered and Eq.(5.70) does not imply (5.69) any more. We can understand it by considering the linear approximation to E,

$$E \simeq E(B_Y^c = 0) - \mu^{(1)} B_Y^{ext} \cos\theta_w \equiv \frac{4\pi v}{g}\left(\varepsilon_0 - \frac{\sin 2\theta_w}{g}\frac{B_Y^{ext}}{v^2} m^{(1)}\right) \qquad (5.81)$$

where $m^{(1)}$ is the $O(\theta_W)$ dipole moment expressed in units of $e/\alpha_W M_W(T)$. From the Fig.5.1 we see that even if $v(T_c)/T_c \geq 1$ the washout condition $E/T_c \geq 35$ is far from being fulfilled.

It follows form the previous considerations, that even if a strong magnetic fields might increase the strength of the EWPT, such an effect would not help baryogenesis.

Conclusions

In this review we have analyzed a large variety of aspects of magnetic fields in the early Universe. Our exposition followed an inverse-chronological order. In the first part of Chap.1 we discussed what observations tell us about recent time fields and their evolution in galaxies and clusters of galaxies. As we have seen, a final answer about the origin of these fields is not yet available. Several arguments, however, suggest that galactic and cluster fields were preexisting, or at least contemporary to, their hosts. The main reasons in favor of this thesis are: the ubiquity of the fields and the uniformity of their strength; the theoretical problems with the MHD amplification mechanisms, especially to explain the origin of cluster fields; the observation of μG magnetic fields in high-redshift galaxies. It is reassuring that new ideas continuously appear to determine these fields at all times. For example, a very recent one by Loeb and Waxman proposes to look for fluctuations in the radio background from intergalactic synchrotron emission of relativistic electrons interacting with CMBR [232].

A consistent and economical mechanism which may naturally explain the early origin of both galactic and cluster magnetic fields, is the adiabatic compression of a primeval field with strength in the range $B_0 \sim 10^{-9} - 10^{-10}$ Gauss (B_0 is the intensity that the primordial field would have today under the assumption of adiabatic decay of the field due to the Hubble expansion). If this was the case, two other interesting effects may arise: *a)* magnetic fields may have affected structure formation perhaps helping to solve some of the problems of the CDM scenario; *b)* magnetic fields can have produced observable imprints in the CMBR. Given the current theoretical uncertainties about the MHD of galaxies and clusters, and the preliminary status of N-body simulations in the presence of magnetic fields, the most promising possibility to test the primordial origin hypothesis of cosmic magnetic fields comes from the forthcoming observations of the CMBR anisotropies.

In Chap.2 we reviewed several possible effects of magnetic fields on the CMBR. In first place, we showed that magnetic fields may affect the isotropy of the CMBR. A magnetic field which is homogeneous through the entire Hubble volume, would spoil Universe isotropy giving rise to a dipole anisotropy in the CMBR. On the basis of this argument it was shown that COBE measurements provide an upper limit on the present time equivalent strength of a homogeneous cosmic magnetic field which is roughly 3×10^{-9} G. More plausibly, magnetic fields are tangled on scales much smaller than the Hubble radius. In this case the effect on the Universe geometry is negligible and much more interesting effects may be produced on small angular scales. Some of these effects arise as a consequence of MHD modes appearing in the magnetized photon-baryon plasma in place of the usual acoustic modes. The amplitude and velocity of the MHD modes depend on the magnetic field intensity and spatial direction. Some of these modes are quite different from standard scalar and tensor modes which are usually considered in the theoretical analysis of the CMBR distortions. For example, Alfvén waves have the peculiar property of not beeing depleted by the Universe expansion inspite of their vectorial nature. These modes are well suited to probe perturbations as those generated by cosmic defects and primordial phase transitions. Tangled magnetic fields, whose production is predicted by several models and, which are observed in the intercluster medium, are also expected to produce Alfvén waves. Another interesting aspect of this kind of isocurvature perturbations, is that they are not affected by Silk damping. The polarization power spectrum of CMBR can also be affected by primordial magnetic fields. This a consequence of the Faraday rotation produced by the field on the CMB photons on their way through the last scattering surface. Magnetic fields with strength $B_0 \gtrsim 10^{-9}$ G may have produced a detectable level of depolarization. Furthermore, it was shown that because of the polarization dependence of the Compton scattering, the depolarization can feed-back into a temperature anisotropy. It was concluded that the best strategy to identify the imprint of primordial magnetic fields on the CMBR is probably to look for their signature in the temperature and polarization anisotropies cross-correlation. This method may probably reach a sensitivity of $\sim 10^{-10}$ G for the present time equivalent magnetic field strength when the forthcoming balloon and satellite missions data is analyzed. Some results from the Boomerang and Maxima experiments [17] are already out with surprising results. It is too early to decide the reasons why, if experimentally confirmed, the second peak is low. We have verified, together with Edsjö [65], that magnetic fields can only decrease the ratio of amplitude of the second peak with respect to the first. Therefore the effect cannot be explained in terms of primordial magnetic fields. Interesting constraints on the strength of these fields will be available only when the amplitude of several peaks and the polarization of CMBR will be measured.

Another period of the Universe history when primordial magnetic fields may have produced observable consequences is the big-bang nucleosynthesis. This subject was treated in Chap.3. Three main effects have been discussed: the effect of the magnetic field energy density on the Universe expansion; the modification produced by a strong magnetic field of the electron-positron gas thermodynamics; the modification of the weak processes keeping neutrons and protons in chemical equilibrium. All these effects produces a variation in the final neutron-to-proton ratio, hence in the relative abundances of light relic elements. The effect of the field on the Universe expansion rate was showed to be globally dominant, though the others cannot be neglected. Furthermore, the non-gravitational effects of the magnetic field can exceed that on the expansion rate in delimited regions where the magnetic field intensity may be larger than the Universe mean value. In this case these effects could have produced fluctuations in baryon to photon ratio and in the relic neutrino temperature. Apparently, the BBN upper bound on primordial magnetic fields, which is $B_0 \lesssim 7 \times 10^{-5}$ G, looks less stringent than other limits which come from the Faraday rotation measurements (RMs) of distant quasars, or from the global isotropy of the CMBR. However, we showed in Sec.3.4 that this conclusion is not correct if magnetic fields are tangled. The reason is that BBN probes length scales which are of the order of the Hubble horizon size at BBN time (which today corresponds approximately to 100 pc) whereas CMBR and the RMs probe much larger scales. Furthermore, constraints derived from the analysis of effects taking place at different times may not be directly comparable if the magnetic field evolution is not adiabatic.

In Chap.4 we reviewed some of the models which predict the generation of magnetic fields in the early Universe. We first discussed those models which invoke a first order phase transition. This kind of transitions naturally provide some conditions, as charge separation, out-of-equilibrium condition, and high level of turbulence, which are known to be important ingredients of magnetogenesis. We discussed the cases of the QCD and the electroweak phase transitions (EWPT). In the case of the EWPT some extension of the particles physics standard model has to be invoked for the transition to be first order. Magnetic fields may be generated during the EWPT from the non-trivial dynamics of the gauge fields produced by the equilibration of the electrically charged components of the Higgs field. This effect resembles the Kibble mechanism for the formation of topological defects. It is interesting that such a mechanism gives rise to magnetic fields, though only on very small scales, even if the phase transition is second order. In general, since the production of magnetic fields during a phase transition is a causal phenomenon, the coherence length scale of these fields at the generation time cannot exceed the horizon radius at that time. Typically, once this length is adiabatically re-scaled to present time, one gets coherence cell sizes which are much smaller than those observed today in galaxies and the inter-cluster medium. This problem may be eased by the effect of the magnetic helicity which is expected to be produced during primordial phase transitions.Helicity may help the formation of large magnetic structures starting from small ones (inverse cascade). Indeed, this is a quite common phenomenon in MHD. Some estimates of the quantitative relevance of this effect have been given,for example, in Sec.1.4. We have seen that the QCD phase transition might indeed give rise to phenomenological interesting values of the present time magnetic field strength and coherence size but only assuming quite optimistic conditions. Magnetic fields produced at the EWPT might have played a role in the generation of galactic magnetic fields only if they were amplified by a galactic dynamo. The problem with the small coherence scale of magnetic fields produced in the early Universe may be circumvented if the production mechanism was not-causal. This may be possible if magnetic fields were produced during inflation by the superadiabatic amplification of preexisting quantum fluctuations of the gauge fields. This phenomenon, however, can take place only if the conformal invariance of the electromagnetic field is broken. In Sec.4.5 we have discussed several interesting mechanisms which have been proposed in the literature to avoid this obstacle. Unfortunately, although some results are encouraging, at the present status of art, none of these model seems to offer any firm prediction. Further work on the subject is, therefore, necessary.

Even if magnetic fields, produced during the electroweak phase transitions or before, are nor the progenitor of galactic magnetic fields, they may still have had other interesting cosmological consequences. Perhaps the most intriguing possibility is that magnetic fields played a role in the generation of the baryon asymmetry of the Universe (BAU). The magnetic fields may influence electroweak baryogenesis at two levels. At a first level, magnetic fields can play an indirect role on electroweak baryogenesis by modifying the free energy difference between the symmetric and broken phases, the Higgs effective potential, and the rate of sphaleron, baryon number violating, transitions. In Chap.5 we showed, however, at this level, no significative modifications to arise with respect to the standard scenario. Magnetic fields, or better their hypermagnetic progenitors, may have played a much more direct role in the generation of the BAU if they possessed a net helicity. Indeed, it is well known from field theory that the hypermagnetic helicity coincides with the Chern-Simon number which can be converted into baryons and leptons numbers by the Abelian anomaly. The origin of the primordial magnetic helicity is still matter of speculation. Among other possibilities which we have review in Chap.4, one of the more discussed in the literature is that a net hypermagnetic helicity of the Universe arise by an anomalous coupling of the gauge fields to an oscillating pseudoscalar field. The existence of pseudoscalar fields of this kind is required by several extensions of the particle physics standard model. However, it must be admitted that the mechanisms for generation of fields that are large and extended at the same time are far from being fully understood.

Large magnetic fields would also have a profound effect on chirality but it is also quite far from observability. As a rule of the thumb, particle physics effects will appear at the earliest at $\mathrm{B} = m_\pi^2$, the π being the lightest hadron. This makes these effects difficult to test. Large fields are required. The only hope is the existence of superconductive strings.

Are the observed fields, so widespread, of early origin or some seeds fields were rapidly enhanced by a dynamo mechanism? This question remains unanswered but high precision CMBR acoustic peak measurements may very well provide a breakthrough.

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